Quadratic Equations

Lessons Overview

- **Introduction to Quadratic Equations**
- **Types of Equations:**
  - Pure
  - Incomplete
  - Perfect Squares
  - Special Trinomials
- **Solving Techniques:**
  - Factoring (ZPP)
  - Completing the Square
  - General Formula
- **Discriminant Cases**
- **Step-by-step Exercises**

Quadratic Equations

Standard Form Equation

A quadratic equation in standard form is written as:

$$ax^2 + bx + c = 0.$$

- a, b, and c are real numbers, with $a \neq 0$ (otherwise it wouldn't be a quadratic equation).
- "Standard form" simply means the terms are ordered by degree.

Example

1. What is the standard form of $4 + x = -x^2$?
2. Is $x^2 + 2 = 0$ in standard form?

Roots of an Equation

Roots of an Equation

To a quadratic equation like $ax^2 + bx + c = 0$, we can associate the polynomial $P(x) = ax^2 + bx + c$.
A solution x_0 is a value that, when substituted into $P(x)$, makes the polynomial equal to zero.

Example

Let's consider the equation $x^2 - 4 = 0$, with associated polynomial $P(x) = x^2 - 4$.
Substituting $x = 2$:

$$P(2) = 2^2 - 4 = 4 - 4 = 0$$

Since $P(2) = 0$, $x = 2$ is a solution of the equation $x^2 - 4 = 0$.

Question: What are the solutions to $x^2 + 2 = 0$?

Square Roots and Quadratic Equation Solutions

Square Root

The square root of a number, such as $\sqrt{9}$, is defined as the positive value whose square gives the original number. For example, $\sqrt{9} = 3$, since $3^2 = 9$. Yet, note that $(-3)^2 = 9$ as well.
We define the square root as the positive value only, to keep the operation well-defined and simple.

Solving Quadratic Equations

When solving an equation like $x^2 = 9$, we look for all numbers whose square is 9.
In this case, there are two solutions: $x = 3$ and $x = -3$, since both 3^2 and $(-3)^2$ equal 9. Unlike the square root, we consider both the positive and negative roots.

Square Roots of Negative Numbers and No Real Solutions

No Real Square Root of Negative Numbers

The square root of a number, such as $\sqrt{81}$, is a number that squared gives the radicand.
However, there is no real square root of a negative number like $\sqrt{-81}$, since no real number squared can yield a negative result.

Quadratic Equations Without Solutions

This means that some quadratic equations have no real solution. For example, the equation $x^2 + 2 = 0$ has no real solutions, because solving it would require computing the square root of -2, which is not a real number.

Number of Solutions of a Quadratic Equation

Number of Solutions

A quadratic equation can have 0, 1, or 2 real solutions:

- 0 if the equation has no real roots,
- 1 if it has a double root,
- 2 if the root involves a positive square root.

Exercise

Determine how many solutions each equation has:

1. $x^2 = 64$
2. $x^2 = -7$
3. $(x + 3)^2 = 0$

Quadratic Equations with One Solution

Perfect Square and Single Solution

A quadratic equation has a unique real solution when the associated polynomial is a perfect square of a binomial. This means it can be written as:

$$(x + a)^2$$

The formula is:

$$(x + a)^2 = x^2 + 2ax + a^2$$

If a quadratic is of this form, it has one solution.

Example

Consider the equation $x^2 + 4x + 4 = 0$.

- This polynomial is the square of $(x + 2)$, since:

$$x^2 + 4x + 4 = (x + 2)^2$$

- Since $(x + 2)^2 = 0$, the only solution is $x = -2$.
- The solution is the value that makes the binomial vanish.

Pure Quadratic Equations

What Are Pure Quadratic Equations?

Pure quadratic equations are a special type of quadratic equation where the linear term is missing. They have the form:

$$ax^2 + c = 0$$

In these equations, the variable appears only as a square. To solve them, just isolate x^2 and find the **two values** that satisfy the equation.

Example: Solving $2x^2 - 8 = 0$

- First, isolate x^2:
$$2x^2 = 8$$
- Then, divide both sides by 2:
$$x^2 = 4$$
- Finally, find the two numbers whose square is 4:
$$x = -2 \vee x = 2 \quad (x = \pm 2)$$

Exercises on Quadratic Equations

Exercises

Solve the following equations:

1. $x^2 - 5 = 0$
2. $4x^2 - 1 = 0$
3. $x^2 + \sqrt{2} - \sqrt{3} = 0$
4. $x^2 - \sqrt{2} + \sqrt{3} = 0$
5. $4x^2 - x(2 - x) + 2(x - 1) = 3$
6. $(x - 1)^2 + (x + 1)^2 = (x + 1)(x - 1) + 5$
7. Which real numbers are equal to their own square?

Solutions to the Exercises

Solutions

1. $x^2 - 5 = 0$
   Solution: $x = \pm\sqrt{5}$
2. $4x^2 - 1 = 0$
   Solution: $x = \pm\frac{1}{2}$
3. $x^2 + \sqrt{2} - \sqrt{3} = 0$
   Solution: $x = \pm\sqrt{\sqrt{3} - \sqrt{2}}$
4. $x^2 - \sqrt{2} + \sqrt{3} = 0$
   No real solution, since $x^2 = \sqrt{2} - \sqrt{3}$ is a square equal to a negative number.
5. $4x^2 - x(2 - x) + 2(x - 1) = 3$
   Simplifies to: $5x^2 - 5 = 3 \Rightarrow 5x^2 = 8 \Rightarrow x = \pm 1$
6. $(x - 1)^2 + (x + 1)^2 = (x + 1)(x - 1) + 5$
   Expands to: $2x^2 + 2 = x^2 + 4 \Rightarrow x^2 = 2$
   Solution: $x = \pm\sqrt{2}$
7. Answer: 0 and 1 are the only real numbers equal to their own square.

ZPP – Zero Product Property

Proposition

If the product of two numbers is zero, then at least one of the two numbers is zero:

$$ab = 0 \Leftrightarrow a = 0 \vee b = 0$$

Proof

Proof by contradiction: if denying the statement leads to a contradiction, then the statement is true.
Suppose $a \neq 0$ and $b \neq 0$, and

$$a \cdot b = 0$$

Divide both sides by a:

$$\frac{1}{a} \cdot a \cdot b = \frac{1}{a} \cdot 0 \Rightarrow b = 0$$

But we assumed $b \neq 0$, which is a contradiction. $\square$

Incomplete Quadratic Equations

What Are Incomplete Quadratic Equations?

Incomplete quadratic equations are equations where the constant term c is missing. They have the form:

$$ax^2 + bx = 0$$

To solve them, we factor out x and apply the ZPP (Zero Product Property).

Example

$x^2 - 4x = 0$
Factor out x:

$$x(x-4) = 0$$

Using ZPP:

$$x = 0 \quad \text{or} \quad x = 4$$

Incomplete Quadratic Equations

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To solve them, we factor out x and apply the ZPP (Zero Product Property).

Example

Consider the equation:

$$x^2 - 4x = 0$$

Factor out x:

$$x(x - 4) = 0$$

Applying the ZPP:

$$x = 0 \quad \text{or} \quad x = 4$$

Exercises – Incomplete Quadratic Equations

Exercises

Solve the following incomplete quadratic equations:

1. $x^2 - 9x = 0$
2. $2x^2 + 6x = 0$
3. $x(x - 5) = 0$
4. $4x^2 - 16x = 0$
5. $x^2 - 2x = 0$

Solutions to the Exercises

Solutions

1. $x^2 - 9x = 0$
   Solution: $x(x-9) = 0 \Rightarrow x = 0$ or $x = 9$ $[x = 0, 9]$
2. $2x^2 + 6x = 0$
   Solution: $2x(x+3) = 0 \Rightarrow x = 0$ or $x = -3$ $[x = 0, -3]$
3. $x(x-5) = 0$
   Solution: $x = 0$ or $x = 5$ $[x = 0, 5]$
4. $4x^2 - 16x = 0$
   Solution: $4x(x-4) = 0 \Rightarrow x = 0$ or $x = 4$ $[x = 0, 4]$
5. $x^2 - 2x = 0$
   Solution: $x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$ $[x = 0, 2]$

Quadratic Equation Exercises

Exercises

Solve the following quadratic equations:

1. $x^2 + 2x + 1 = 0$
2. $x^2 + 6x + 9 = 0$
3. $x^2 + 12x + 36 = 0$
4. $x^2 - 16x + 64 = 0$
5. $4x^2 + 20x + 25 = 0$
6. $9x^2 - 54x + 81 = 0$

Solutions to the Exercises

Solutions

1. $x^2 + 2x + 1 = 0$ $(x + 1)^2 = 0 \Rightarrow x = -1$
2. $x^2 + 6x + 9 = 0$ $(x + 3)^2 = 0 \Rightarrow x = -3$
3. $x^2 + 12x + 36 = 0$ $(x + 6)^2 = 0 \Rightarrow x = -6$
4. $x^2 - 16x + 64 = 0$ $(x - 8)^2 = 0 \Rightarrow x = 8$
5. $4x^2 + 20x + 25 = 0$ $(2x + 5)^2 = 0 \Rightarrow x = -\frac{5}{2}$
6. $9x^2 - 54x + 81 = 0$ $(3x - 9)^2 = 0 \Rightarrow x = 3$

Quadratics Solvable via Binomial Square

Exercises

Solve the following equations:

1. $x^2 + 2x + 1 = 0$
2. $x^2 + 6x + 9 = 0$
3. $x^2 + 12x + 36 = 0$
4. $x^2 - 16x + 64 = 0$
5. $4x^2 + 20x + 25 = 0$
6. $9x^2 - 54x + 81 = 0$

Solutions to the Exercises

Solutions

1. $x^2 + 2x + 1 = 0$ $[x = -1]$
2. $x^2 + 6x + 9 = 0$ $[x = -3]$
3. $x^2 + 12x + 36 = 0$ $[x = -6]$
4. $x^2 - 16x + 64 = 0$ $[x = 8]$
5. $4x^2 + 20x + 25 = 0$ $[x = -\frac{5}{2}]$
6. $9x^2 - 54x + 81 = 0$ $[x = 3]$

Partial Factoring

Partial Factoring

In a polynomial, we group two pairs of terms and factor each group separately:

$$ax + ay + bx + by = x(a+b) + y(a+b)$$

Then we factor out the common binomial:

$$(a+b)(x+y)$$

Example

Let's solve the equation $2x^2 + 4x + 3x + 6 = 0$ using partial factoring and the Zero Product Property (ZPP):

$$2x(x+2) + 3(x+2) = 0$$

Factor $(x+2)$:

$$(2x+3)(x+2) = 0$$

Apply ZPP:

$$2x + 3 = 0 \quad \text{or} \quad x + 2 = 0$$

The solutions are: $x = -\frac{3}{2} \quad \text{or} \quad x = -2$

Special Trinomials

Special Trinomial

A polynomial $P(x) = ax^2 + bx + c$ such that $a = 1$, $b = n_1 + n_2$, and $c = n_1 n_2$, with $n_1, n_2 \in \mathbb{Z}$, is called a special trinomial and can be written as:

$$P(x) = (x + n_1)(x + n_2)$$

Example

An equation containing this type of polynomial can be solved using the Zero Product Property. For example, the equation $x^2 + 7x + 12 = 0$ becomes:

$$(x + 3)(x + 4) = 0$$

From which the solutions are $x = -3$ and $x = -4$.

Proof of the Special Trinomial Formula

Where the Formula Comes From

Let's consider a special trinomial of the form $P(x) = x^2 + sx + p$, where $s = a + b$ and $p = a \cdot b$. We can factor this trinomial using partial factoring. Start by writing the trinomial as a sum of four terms:

$$P(x) = x^2 + ax + bx + ab$$

Now group the terms into two pairs and factor each group:

$$x(x + a) + b(x + a)$$

Notice that both expressions contain the common factor $(x + a)$, which we can factor out:

$$(x + a)(x + b)$$

This proves that the special trinomial $x^2 + sx + p$ can be factored as the product of $(x + a)$ and $(x + b)$.

Special Trinomial Exercises

Exercises

Solve the following quadratic equations:

1. $x^2 + 3x + 2 = 0$
2. $x^2 + 5x + 6 = 0$
3. $x^2 - 7x + 12 = 0$
4. $x^2 - 2x - 15 = 0$
5. $x^2 + 6x - 16 = 0$
6. $x^2 - 5x - 24 = 0$

Special Trinomial Exercises - Solutions

Solutions

1. $x^2 + 3x + 2 = 0$ $\quad x_1 = -1,\ x_2 = -2$
2. $x^2 + 5x + 6 = 0$ $\quad x_1 = -2,\ x_2 = -3$
3. $x^2 - 7x + 12 = 0$ $\quad x_1 = 3,\ x_2 = 4$
4. $x^2 - 2x - 15 = 0$ $\quad x_1 = 5,\ x_2 = -3$
5. $x^2 + 6x - 16 = 0$ $\quad x_1 = 2,\ x_2 = -8$
6. $x^2 - 5x - 24 = 0$ $\quad x_1 = 8,\ x_2 = -3$

Methods for Solving Quadratic Equations

Summary of Methods Covered So Far

Equation Type	**Solving Method**	**Example**
Pure Quadratics	Isolate x^2 and extract the square root to get **two** solutions.	$2x^2 = 18$ $\Rightarrow x^2 = 9$ $\Rightarrow x = \pm 3$
Spurious Quadratics	Factoring and ZPP.	$x^2 - 5x = 0$ $x(x-5) = 0$ $\Rightarrow x = 0$ or $x = 5$
Binomial Square	Recognize the square of a binomial and solve. Has **one** solution.	$x^2 + 8x + 16$ $(x+4)^2 = 0$ $\Rightarrow x = -4$
Special Trinomial	Factoring and ZPP.	$x^2 + x - 6 = 0$ $(x+3)(x-2) = 0$ $\Rightarrow x = -3$ or $x = 2$

Solving $(3x-5)^2 = 16$

Step-by-Step Solution

1. Start by taking the square root of both sides:

$$(3x-5)^2 = 16 \Rightarrow 3x-5 = \pm\sqrt{16} = \pm 4$$

2. Solve both resulting equations:

$$1.\ 3x-5=4 \quad \text{and} \quad 2.\ 3x-5=-4$$

First:

$$3x = 9 \Rightarrow x = 3$$

Second:

$$3x = 1 \Rightarrow x = \frac{1}{3}$$

3. Final solutions:

$$x_1 = 3 \quad \text{and} \quad x_2 = \frac{1}{3}$$

Exercises

Solve the following equations:

1. $(3x-5)^2 = 16$
2. $(x+7)^2 = 9$
3. $\left(\frac{1}{3} - \frac{1}{2}x\right)^2 = \frac{1}{4}$
4. $(\sqrt{2}x-1)^2 - 32 = 0$
5. $\left(-5x - \frac{2}{3}\right)^2 = \frac{1}{9}$

Solutions

Solutions

1. $(3x-5)^2 = 16 \Rightarrow x = 3$ or $x = \frac{1}{3}$
2. $(x+7)^2 = 9 \Rightarrow x = -4$ or $x = -10$
3. $\left(\frac{1}{3} - \frac{1}{2}x\right)^2 = \frac{1}{4} \Rightarrow x = -\frac{1}{3}$ or $x = \frac{5}{3}$
4. $(\sqrt{2}x - 1)^2 = 32 \Rightarrow x = 3$ or $x = -1$
5. $\left(-5x - \frac{2}{3}\right)^2 = \frac{1}{9} \Rightarrow x = -\frac{1}{5}$ or $x = \frac{1}{15}$

Completing the Square

How do we solve $x^2 + 4x - 6 = 0$?

The goal is to transform the equation $x^2 + 4x - 6 = 0$ into a solvable form by completing the square, that is, rewriting $x^2 + 4x$ as a perfect square.

1. Identify the constant needed to complete the square. For $x^2 + 4x$, we need 4:
$$(x+2)^2 = x^2 + 4x + 4$$
2. Add and subtract 4 within the equation:
$$x^2 + 4x - 6 = (x^2 + 4x + 4) - 4 - 6 = (x+2)^2 - 10$$
3. Solve:
$$(x+2)^2 = 10 \Rightarrow x + 2 = \pm\sqrt{10}$$
$$x = -2 \pm \sqrt{10}$$

Final solutions: $x = -2 + \sqrt{10}$ and $x = -2 - \sqrt{10}$

Completing the Square

How do we solve $x^2 + 4x - 6 = 0$?

The goal is to transform the equation $x^2 + 4x - 6 = 0$ into a form that's easier to solve. We do this by completing the square, rewriting the trinomial $x^2 + 4x$ as a perfect square.

1. Consider $x^2 + 4x$ and identify the constant term needed to complete the square. For $x^2 + 4x$, the missing term is 4 because:

$$(x+2)^2 = x^2 + 4x + 4$$

2. Add and subtract 4 inside the equation:

$$x^2 + 4x - 6 = (x^2 + 4x + 4) - 4 - 6 = (x+2)^2 - 10$$

3. Solve the resulting equation:

$$(x+2)^2 - 10 = 0 \quad \Rightarrow \quad (x+2)^2 = 10$$

4. Take the square root:

$$x + 2 = \pm\sqrt{10} \quad \Rightarrow \quad x = -2 \pm \sqrt{10}$$

General Completing the Square Formula

Formula

In general, to complete the square for an expression of the form $x^2 + bx$, we add and subtract $\left(\frac{b}{2}\right)^2$:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Example: Solve $x^2 + 6x + 1 = 0$ by completing the square

Identify the coefficient of x, which is 6. Complete the square by adding and subtracting $\left(\frac{6}{2}\right)^2 = 9$:

$$x^2 + 6x + 1 = (x^2 + 6x + 9) - 9 + 1 = (x+3)^2 - 8$$

So:

$$(x+3)^2 - 8 = 0 \quad \Rightarrow \quad (x+3)^2 = 8$$

$$x + 3 = \pm\sqrt{8} \quad \Rightarrow \quad x = -3 \pm \sqrt{8}$$

Final solutions:

Exercises – Completing the Square

Exercises

Solve the equations by completing the square:

1. $x^2 + 6x + 2 = 0$
2. $x^2 - 4x - 5 = 0$
3. $x^2 + 8x + 3 = 0$
4. $x^2 - 10x + 4 = 0$

Solutions – Completing the Square

Solutions

1. $x^2 + 6x + 2 = 0$
   $(x+3)^2 - 7 = 0 \quad \Rightarrow x = -3 \pm \sqrt{7}$
2. $x^2 - 4x - 5 = 0$
   $(x-2)^2 - 9 = 0 \quad \Rightarrow x = 2 \pm 3 \Rightarrow x = 5, -1$
3. $x^2 + 8x + 3 = 0$
   $(x+4)^2 - 13 = 0 \quad \Rightarrow x = -4 \pm \sqrt{13}$
4. $x^2 - 10x + 4 = 0$
   $(x-5)^2 - 21 = 0 \quad \Rightarrow x = 5 \pm \sqrt{21}$

From Completing the Square to the General Formula

Introduction

We've learned how to solve a quadratic equation by completing the square. Now let's see how this method leads to a general formula that works for any quadratic equation of the form $ax^2 + bx + c = 0$.

General Formula

By completing the square, we derive the general formula to solve any quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the **quadratic formula**, which gives the solutions of $ax^2 + bx + c = 0$.

The General Formula

Consider the equation $ax^2 + bx + c = 0$, with $a \neq 0$. Divide by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Complete the square:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Before taking the square root, we analyze the right side. Since $4a^2 > 0$, we focus on the term $b^2 - 4ac$, which is called the **discriminant** and denoted by Δ.

The General Formula: The Discriminant and Solutions

Now let's examine the discriminant $\Delta = b^2 - 4ac$ and the resulting cases:

- **If** $\Delta > 0$: The equation has two distinct real solutions.

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

- **If** $\Delta = 0$: The equation has one real (repeated) solution.

$$x = -\frac{b}{2a}$$

- **If** $\Delta < 0$: The equation has no real solutions.

Exercises

Solve the following equations:

1. $2x^2 + 3x = -1$
2. $-x^2 = x - 12$
3. $2x^2 - 5 = -4x$
4. $2(x-1) = 3x^2$
5. $x(x-2) = 15$
6. $x(x-5) = 24$
7. $2(x+3)(x+1) = x + 15$

Solutions

Solutions

1. $\left\{-\frac{1}{2}, -1\right\}$
2. $\{-4, 3\}$
3. $\left\{-2 \pm \frac{\sqrt{14}}{2}\right\}$
4. No real solutions
5. $\{-3, 5\}$
6. $\{-3, 8\}$
7. $\left\{\frac{9}{2}, 1\right\}$

Advanced Equation

$$x^2 - (\sqrt{3} - \sqrt{2})x - \sqrt{6} = 0$$

Solving the Equation

Solution

Given:

$$x^2 - (\sqrt{3} - \sqrt{2})x - \sqrt{6} = 0$$

with $a = 1$, $b = -(\sqrt{3} - \sqrt{2})$, $c = -\sqrt{6}$
Compute the discriminant:

$$\Delta = b^2 - 4ac = (\sqrt{3} - \sqrt{2})^2 + 4\sqrt{6}$$

$$b^2 = 3 + 2 - 2\sqrt{6}, \quad -4ac = 4\sqrt{6}$$

$$\Delta = 5 - 2\sqrt{6} + 4\sqrt{6} = 5 + 2\sqrt{6} = (\sqrt{3} + \sqrt{2})^2$$

So:

$$x = \frac{-b \pm \sqrt{\Delta}}{2} = \frac{\sqrt{3} - \sqrt{2} \pm (\sqrt{3} + \sqrt{2})}{2}$$

Thus:

$$x_1 = \sqrt{3}, \quad x_2 = -\sqrt{2}$$

Conclusion

Key Takeaways

- Quadratic equations appear in various forms and can be solved with different techniques.
- Recognizing the type of equation helps choose the most efficient method:
  - Factoring
  - Completing the square
  - Using the general formula
- The discriminant gives information about number and nature of the solutions.

What's Next?

Practice is key!