An Informal Introduction to Function Limits

Cesare Peli

Finite limit as x approaches a finite value

Intuitive Definition

The limit of a function $f(x)$ at a point $x = x_0$ describes the value that the function approaches as x gets closer to x_0.
If $f(x)$ gets closer and closer to a value L as x approaches x_0 **from both sides**, then we say that the limit of $f(x)$ as x approaches x_0 is L, and we write:

$$\lim_{x \to x_0} f(x) = L$$

Finite limit as x approaches a finite value

Example

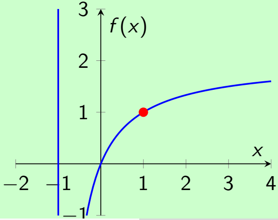
Consider the function $f(x) = \frac{2x}{x+1}$.

$$\lim_{x \to 1} f(x) = \frac{2 \cdot 1}{1 + 1} = \frac{2}{2} = 1$$

Here, the function approaches the value 1 as x gets close to 1.

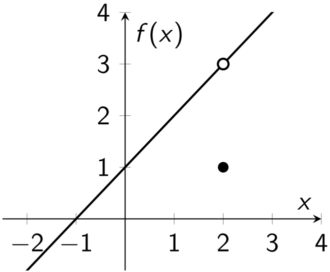
Graphical Interpretation

Looking at the graph of $f(x) = \frac{2x}{x+1}$, we can visually see that the function gets closer to the value 1 as x approaches 1.



Limit and Continuity at a Point

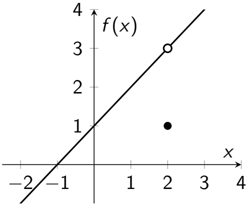
Let $f(x) = x + 1$. We can notice that there is a discontinuity at the point $(2, 3)$, while the point $(2, 1)$ **is** included in the graph of the function.



Limit Evaluation

- $\lim_{x \to -1} f(x) = ?$
- $\lim_{x \to 0} f(x) = ?$
- $\lim_{x \to 2} f(x) = ?$

Limit and Continuity at a Point



Limit Results

- $\lim_{x \to -1} f(x) = 0$
- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 2} f(x) = 3$

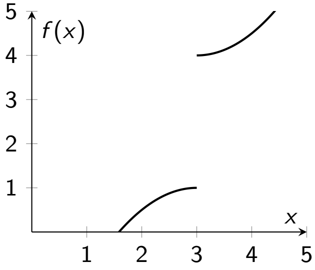
Important Note

We are not asking for the value of the function at $x = 2$, but for the value the function **approaches** as x gets infinitely close to 2.

Left-hand and Right-hand Limits

Let's consider the function $f(x)$, which is defined piecewise:

$$f(x) = \begin{cases} -\frac{1}{2}(x-3)^2 + 1 & \text{if } x \leq 3, \\ \frac{1}{2}(x-3)^2 + 4 & \text{if } x > 3. \end{cases}$$



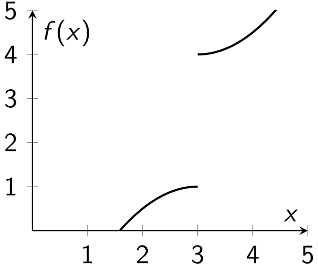
Limit Evaluation

- $\lim_{x \to 3^-} f(x) = ?$
- $\lim_{x \to 3^+} f(x) = ?$
- $\lim_{x \to 3} f(x) = ?$

Left-hand and Right-hand Limits

Limit Evaluation

- $\lim_{x \to 3^-} f(x) = 1$
- $\lim_{x \to 3^+} f(x) = 4$

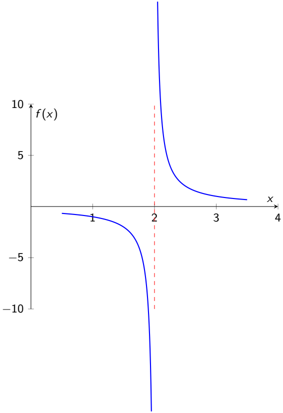


The Limit Does Not Exist

The limit $\lim_{x \to 3} f(x)$ does not exist, because by definition **a limit exists at a point only if the left-hand and right-hand limits are equal**. This is another example of a discontinuity.

Vertical Asymptotes

Let's consider the function $f(x) = \frac{1}{x-2}$, which has a vertical asymptote at $x = 2$.



Limit Evaluation

- $\lim_{x \to 2^-} f(x) = +\infty$
- $\lim_{x \to 2^+} f(x) = -\infty$

Intro to Limit Arithmetic: Infinite Limits

Why does a constant divided by infinity approach zero?

Imagine dividing a constant, such as 1 or 5, by increasingly large numbers, like 1000, 1000000, or 1000000000. As the denominator increases, the result gets smaller:

$$\frac{1}{1000} = 0.001, \quad \frac{1}{1000000} = 0.000001, \quad \text{and so on.}$$

Eventually, as the denominator tends to infinity, the result gets arbitrarily close to 0. That's why $\frac{k}{\infty} = 0$, where k is a constant.

Why does something divided by zero tend to infinity?

Now imagine dividing a constant, like 1, by smaller and smaller numbers: 0.1, 0.01, 0.001, etc. As the denominator gets closer to zero, the result grows rapidly:

$$\frac{1}{0.1} = 10, \quad \frac{1}{0.01} = 100, \quad \frac{1}{0.001} = 1000, \quad \text{and so on.}$$

As the denominator becomes extremely small, the result becomes extremely large, tending to $+\infty$ (if the numerator is positive) or $-\infty$ (if negative). That's why we informally write $\frac{k}{0} = \infty$, where k is a constant.

Disclaimer: What Is Infinity?

Infinity Is Not a Number

Infinity is not a number. It's a mathematical concept that represents something unbounded, limitless, or without end. We cannot treat infinity like a regular number: we can't add, subtract, multiply, or divide it the same way we do with finite values.

Common Mistakes and Misuses of Infinity

Here are some common misconceptions:

- Treating infinity as a really big number: expressions like $\infty + 1$ or $\infty \times 2$ don't have real meaning in mathematics.
- Thinking that there is a "bigger" infinity: while math does define different types of infinity, they cannot be compared like ordinary numbers.
- *Dividing by zero*: when the denominator *approaches* zero, the result may tend toward infinity, but this is just a way of describing behavior — it doesn't mean infinity is an actual value.

Vertical and Horizontal Asymptotes

An asymptote is a line that a function gets closer and closer to, but never actually reaches. There are two types:

Vertical Asymptotes

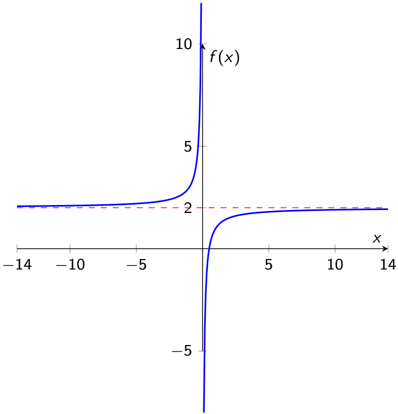
Vertical asymptotes occur when a function tends toward $+\infty$ or $-\infty$ as it approaches a specific point $x = c$. We saw this with the function $f(x) = \frac{1}{x-2}$, which has a vertical asymptote at $x = 2$.

Horizontal Asymptotes

Horizontal asymptotes describe how a function behaves as x approaches $+\infty$ or $-\infty$. A horizontal asymptote exists when $\lim_{x \to \pm\infty} f(x)$ equals a finite value.

Horizontal and Vertical Asymptotes

Consider the function $f(x) = 2 - \dfrac{1}{x}$.



$$\lim_{x \to -\infty} f(x) = \,? \quad \text{and} \quad \lim_{x \to +\infty} f(x) = \,?$$

$$\lim_{x \to 0^+} f(x) = \,? \quad \text{and} \quad \lim_{x \to 0^-} f(x) = \,?$$