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Thesis

GALILEO AND PARABOLIC MOTION, ANALYSING ROLES OF MATHEMATICS IN PHYSICS FOR EDUCATION

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Introduction

This thesis is part of the European project IDENTITIES, wich involves the University of Bologna, Parma, Crete, Montpellier and Barcelona. The name stands for: Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM. Its purpose is to produce didactic modules for teachers and pre-service teachers.

This thesis is situated in the border between the research in mathematics and physics education and aims to contribute to the debate in interdisciplinarity between the two disciplines in a for preservice teacher education. In particular, the thesis is focused on the concept of proof and reports the results of the implementation of a teaching module on parabola and parabolic motion, that includes a special focus on the historical case of Galileo Galileo's demonstration of parabolic motion.

CHAPTER ONE is a introduction to some crucial historical moments concerning the proof, with the aim of pointing out the ideas on proof that Galileo and Newton imported in physics and their role in the scientific revolution. To give an idea of the scope and significance of the scientific revolution we present some characteristics of medieval cosmology, still linked to the Aristotelian conception. We try to see if the transition to modern science is linked or not to a new idea of proof. The results of the analysis led us to accept that there is no an absolute idea of proof, but this is always relative to specific contexts. In order to support this thesis we consider what a proof is in formal terms, that is, for mathematical logic, and whether the proof in mathematics can be only grounded in ancient Greece.

CHAPTER TWO is a review of literature in the field of physics education. The aim is to highlight the different roles of mathematics in the teaching of physics. We discuss, among others, articles by Uhden, Karam and Redish. We reflect on the difference between mathematical proof and experimental verification. We consider different didactic paths in which the epistemological and structural role of mathematics is emphasized. In particular, we discuss articles that show and illustrate examples in which formal relations can be used to structure and develop reasoning.

In CHAPTER THREE we present the IDENTITIES didactic module on parabola and parabolic motion, designed for preservice teachers, that is to mathematics and physics students following a didactic curriculum. In the module a special role is given to Galileo's proof of parabolic motion, which is historically introduced and compared with a demonstration taken from a current textbook. The module also aims to guide the preservice teachers to reflect on the different characteristics and functions that a proof can assume. In various respects, the module contains a synthesis of the themes that have been further explored in chapter one and chapter two, relab in a more effective way from a point of view.

CHAPTER FOUR presents the analysis of the module implementation, held at the University of Bologna in 2020 to students of mathematics and physics. Students were asked to produce written essays and to answer questionnaires, during and at the end of the course. The data was also analyzed to check to what extent future mathematics and physics teachers were are familiar with the concept of proof. In particular, the awareness of the characteristics that define a proof and the awareness of the role of mathematical proof in physics and in a didactic context of physics were tested.

Chapter 1

Introduction to the proof

1.1 A proof in formal terms

In this chapter we want to mention some historical moments in which proof entered the world of mathematics and then of science. it is difficult to give an unambiguous definition of what a proof is. The validity of a proof has always been relative to the context. Let us start considering how even in the field of mathematical logic it is not obvious to have a simple idea of proof.
Before proceeding in this direction, however, we want to pause for a moment by asking ourselves what a proof in mathematics is. Since we want to pose this question in the most general terms possible, we present the point of view of formal logic. The purpose of this example is to show that even choosing the perspective that is least compromised with assumptions of any kind, the result is not entirely satisfactory. So let's see what a proof is in terms of mathematical logic.

In modern mathematics, a theorem is a formal statement of the type

if T then A.

Since the modern mathematics is based on the axiomatic method, we do not consider T as the hypothesis and A as the thesis, but T as the axioms of the theory and A as a statement that can itself be conditional.
For example, T could be the union of field axioms, order axioms and the completeness axiom and A could be the statement:

"If a real-valued function f is continuous on a proper closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that
$f'(c) = 0$". [1]

Let us make a first remark. Let us note that the meanings of the words "function", "continuous function", "differentiable function", "interval", "closed interval", "open interval", are not contained in the statement. In principle these words

[1]Rolle's Theorem.

could mean anything. Which means that the pleasant and concise logical expression

$$T \Rightarrow A$$

does not contain all that is necessary for the verification of its veracity. A precise number of definitions are missing, definitions that must be not arbitrary and supposedly shared. These definitions are the result of a technical, not a logical, construction work.
Let's take a nicer example that is not so technical but more intuitive: we take the axioms of Euclidean geometry as T and the Pythagoras's theorem as A. Also in this case, we would need the definitions of "triangle", "square", "right angle", "hypotenuse". Without these definitions the proposition does not have a unique meaning in itself, it only has it if it is found in, for example, a geometry book.

If we added all the necessary definitions to the statement, the proposition would have a complete sense from the logical point of view, without giving anything implied. Of course we are going to extremes, probably no one has ever done this and no one ever will because mathematical practice does not correspond perfectly to the philosophy of mathematics. It is legitimate to consider a theorem as a simple logical proposition, but we must not forget all the technical tools that mathematics constructs to produce other mathematics. Mathematics as a discipline, not as an abstraction, is a practice.

Now a second remark. In the case of Rolle's theorem, starting from the field axioms is not really rigorous. We have used axioms of field, order, but these are concepts that have no meaning if we are not operating on a set. But what a set is? Unfortunately, unlike other mathematical objects, it is not possible to give a simple definition of a set that does not imply terrible paradoxes[2], unless we give an empty and circular definition such as that of "collection". Sets must be constructed in an axiomatic way and their construction, in order to have a mathematics that works without problems, is not at all trivial and touches delicate and unintuitive aspects such as that of the Axiom of choice. But be rigorous (and here it is a substantial rigor, it is not a question of love of form) it would be necessary to start from the Zermelo-Frankel axioms, construct the real numbers, prove their good ordering, their completeness, construct the usual tools of analysis (the necessary definitions implicitly attached to the statement) and only then prove Rolle's theorem. That is something that no one does even in a university course of analysis. So even in the temples of rigor, the university departments of mathematics, practicality is preferred to logical rigor, and certain questions are intentionally left out, relegated to the limbo of "foundations", which real mathematicians prefer not to deal with.

However, as we have already understood, saying "if T then A" is equivalent to saying that A is a theorem of T. In the modern logical notation, is indicated with:

[2] The naive (and natural) definition of a set, identifying it with the characterization of its elements, can lead to paradoxes linked to self-reference, like most of the paradoxes that afflict mathematical logic. It is the case of the set of sets that do not belong to themselves. If $A = \{x | x \notin A\}$, then $A \in A \Leftrightarrow A \notin A$. This paradox was published by Bertrand Russell in 1903 in *The Principles of Mathematics*.

$$T \models A$$

This can also be read as "if T is true then A is true". But... what does "true" mean? If A is a logical consequence of T, what is a logical consequence? Here we have a problem. We do not really want to deal with these themes here, but let us consider the words of the logician Gabriele Lolli, to get an idea:

> To tell the truth about a knowledge domain, it is necessary to use a language that speaks about the language in which that knowledge is formulated - which is called object language - and to assume other knowledge that can demonstrate properties related to the language of the object and its meanings. [...] No truth is definable at all, but even the truth in a limited domain, without passing to a higher domain, is not definable [3]

Let us consider a fascinating expression, often used by non-mathematicians to express the concreteness that would characterize mathematics:

$$2 + 2 = 4$$

Let us assume to consider the set of natural number $\mathbb{N}$, just to avoid tricks related to finite fields [4]. We know that the expression is true, but we cannot prove it in the domain of arithmetic. We need to move into set theory and in that context formulate a theorem that has this expression as conclusion. Otherwise:

if $2 = 0^{++}$ and $4 = 2^{++} = 0^{++++}$
and $+$ is a injective function
and $0 \neq x+$ and $x + y^{+} = (x + y)^{+}+$ and $x + 0 = x$
than $0^{++} + 0^{++} = 0^{++++}$

Whether or not A is a theorem of T is a fact, but only within an abstract linguistic universe, made up of symbols that could have different interpretations. In principle we cannot exclude an interpretation where the axioms of T are true and A is not.

So, what is a proof? A proof is a guarantee certificate of $T \models A$. It is not possible to give a more precise definition. It doesn't need to be convincing or easy to understand, the only condition is the finitude of the argument. The belief that one can have about the veracity of a demonstration is related to one's maturity and experience, it is not something objective. In other words, an exact truth can exist only within a formal language, but when we give meaning to a formal language, for example a mathematical meaning, the truth is no longer an objective datum but a convention, even if determined by rigorous logical steps. We can prove that

$$2 + 2 = 4$$

[3]Lolli, 2005, p. 14.

[4]For example, in $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, $2 + 2 = 0$.

by treating these objects as symbols, but if we think of them as quantities (two apples plus two apples) we can find the above proof convincing and think that it also concerns our apples or numerosity in general, we would all probably agree, but not we are in the context of an objective truth. In fact we all work with numbers and we often consider them the apotheosis of objectivity but we would be rather embarrassed if we had to explain what numbers are[5].

[5] A first answer might be that numbers are a representation of the abstract concept of quantity. But not all numbers seem to fit this definition, such as complex numbers. We could say that some numeric sets are are a construction that starts from numbers that represent abstract concepts of quantity and become something else. However, if we tried to clarify what a "representation of the abstract concept of quantity" is, we would leave the sphere of mathematics and its foundations and enter directly into that of philosophy. The more mathematical answer to the initial question "what are numbers?", although it may seem a bit rude, is that we don't care what they are, as long as they respect certain properties.

1.2 Is the birth of proof a "Greek miracle"?

In answering questions such as: "Who are the mathematicians that first used proofs as we conceive them today? Who set the canons of the proof in logical terms?" we are likely to come up with things like Plato and Aristotle, Greek culture as the cradle of civilization, the beacon of reason that illuminates the darkness of superstition. All correct concepts, but which can lead to excessive simplifications. Let us consider that:

1. In ancient Greek, mathematical demonstration made its first appearance in *Euclid's Elements* (4th century BC - 3rd century BC) and then in the geometric works of Archimedes (c. 287 BC - c. 212 BC) and Apollonius of Perga (c. 240 BC - c. 190 BC).

2. The canons of deductive and then inductive reasoning were established by Aristotle (384 BC-322 BC) in *Prior Analytics* and *Posterior Analytics*.

3. Modern mathematics and modern philosophy have shown the existence of errors and gaps in these ancient works. Nevertheless they have represented the reference for over two thousand years and form the basis of Western rationality, which constitutes a unique case in the history of universal thought.

Therefore it seems that in pre-Greek mathematics there is not evidence of demonstrations; only in Europe there we see use of mathematics unrelated to the field of applications. Rather than the lack of proofs in pre-Greek mathematics, we should speak of absence of a general theory that justifies the correctness of a proposition. This could be identified as a specificity of Greek thought, wanting to preserve one.

In 2012 Karine Chemla has edited *The History of Mathematical Proof in Ancient Tradition*[6], an important book on Greek mathematics, which contains the interesting essay *Historiography and history of mathematical proof: a research programme.* Here the author reflects on how an exasperated dichotomy has been created between Greek mathematics and "Eastern" mathematics and suggests in what direction it should be overcome.

As Chemla points out, since ancient times, translations of the Elements have been circulating in Greek, Arabic, Latin, Hebrew and then in several European popular vernaculars, constituting the central piece of mathematical education. The proof in those editions constitute a model of incontrovertibility that inspired many philosophers. We therefore also have a non-mathematical use of mathematical proofs, which constitute the foundation, even from an identitary point of view, of Western knowledge. The presence of proofs in Greek mathematics has been used to support its superiority over Arabic, Chinese, Indian, Babylonian, and Egyptian ones.
But nineteenth-century scholars of history of mathematics have found traces of

[6]Chemla, 2012.

proofs in pre-Greek mathematics. Nonetheless these studies have not been sufficient to change the dominate paradigm.

In 1841, Jean-Baptiste Biot write:

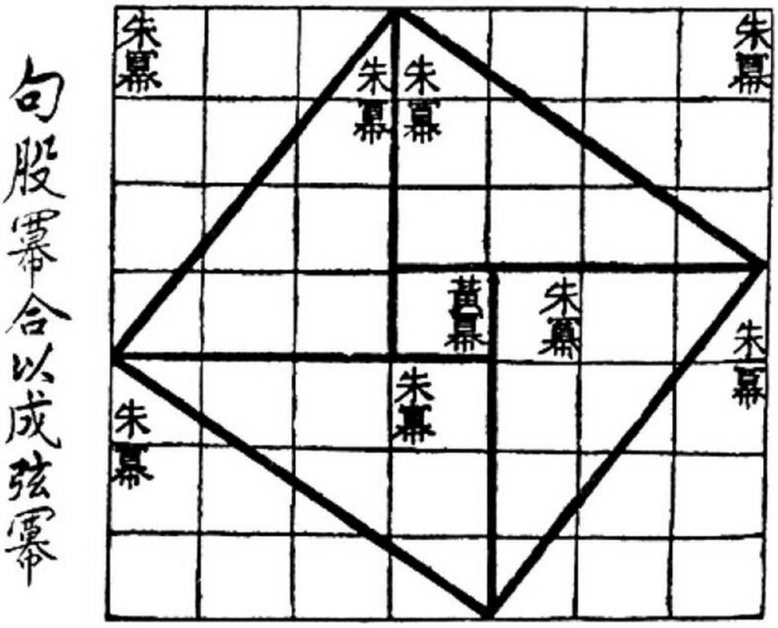
> This peculiar habit of mind, following which the Arabs, as the Chinese and Hindus, limited their scientific writings to the statement of a series of rules, which, once given, ought only to be verified by their applications, without requiring any logical demonstration or connections between them: this gives those Oriental nations a remarkable character of dissimilarity, I would even add of intellectual inferiority, comparatively to the Greeks, with whom any proposition is established by reasoning, and generates logically deduced consequences.[7]

Among the 19th century studies on non-European mathematics, Henry Thomas Colebrook, mathematician and orientalist, is the first to translate Sanskrit mathematics, publishing among others in 1817 *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhāscara*.
Colebrook there encountered a kind of algebraic analysis, with symbols and literal signs. It should be noted that, when Colebrook wrote, analysis it is not yet a field in which rigour plays a central role (we will return to this further on). Colebrook showed the proof of algebraic rules by geometric tools. As François Charette pointed out, several nineteenth-century studies find the presence of proofs in Indian, Chinese, Egyptian and Arabic texts. Some scholars of the same period point out that these proofs are inferior to those of the Elements, while for others they represented alternative demonstrations, in which the lack of a rigor comparable to the Greek one was not enough to not consider them proofs, even interesting.

Like some of the Babylonian tablets, the earliest Chinese writings attest mathematical activity composed of problems and algorithms solving them. The practice of proof to which they bear witness also aims at establishing the correctness of algorithms. Chinese sources demonstrate the fact that operations, called meta-operations, are sometimes applied to the sequence of operations that an algorithm constitutes. These meta-operations were used to transform an algorithm known to be true, called qua algorithm, into another algorithm, whose correctness was to be established.

With the publication of the third volume of Joseph Needham's Science and Civilization in China, in 1954, a line of study was opened. The provocative monograph *Was Pythagoras Chinese?*[8] By Frank Swetz and T.I. Kao presents the important Chinese mathematics text *Chiu chang suan shu*, which also contains a proof of Pythagoras's theorem (of the theorem we know by this name). The dating of the *Chiu chang* is controversial, but it is accepted that much of the material in the book dates back to the time of Confucius, of the sixth century BC, and that the content reflects the mathematical knowledge accumulated until then.

[7] Chemla, 2012, p. 6.
[8] Swetz, 1977.



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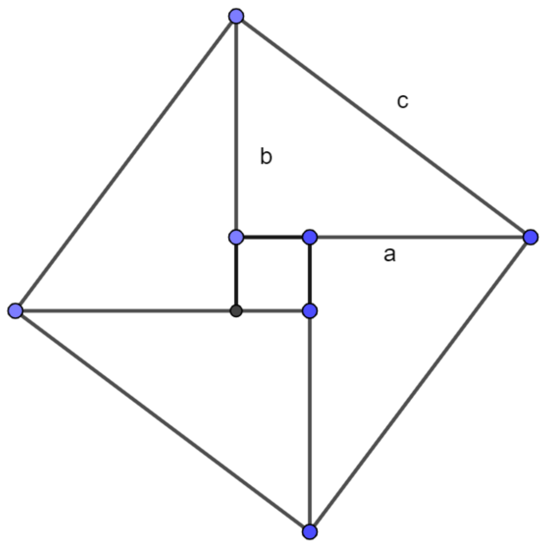


We can see in the figure four congruent right-angled triangles, and let's consider one in particular. We call c the hypotenuse, a the major cathetus and b the minor cathetus. Let's consider the area of the small inner square, this is $(a-b)^2$. The four triangles can be composed to form two rectangles of area ab. So considering the area of the whole square, this is:

$$c^2 = 2ab + (a-b)^2 = 2ab + a^2 - 2ab + b^2 = a^2 + b^2$$

Let us go back to the main theme. Chemla wonders what implications these lines of research bring to the history of proof, and she leaves some questions open:

[9]This diagram is known in China as *hsuan-thu*. Picture from Ralph H. Abraham, 25 April 1996.

> On the one hand, proving is an activity that takes place in specific social and professional groups which have specific agendas.
>
> Indeed, only along these lines can we hope to bring to light and accommodate the variety of practices in a way more satisfactory than the old model of competing civilizations which has been pre-eminent from the nineteenth century onwards.
>
> We have seen that some proofs seem to be conducted in order to understand the statement proved or the text which states it. In other cases, proofs have appeared to have had as one of their goals the identifi cation of fundamental operations or the display of a technique. We have also seen that in some contexts, proofs were expected to be general or to comply with an ideal of generality. In others, they should bring clarity, yield fruitfulness or manifest simplicity. Much more remains to be done in identifying goals and values practitioners have attached – and still attach.[10]

In the perspective of our work, these reflections highlight the cultural aspect of proof, claimed in retrospect as the milestone of a civilization. To claim the title of proof, is it enough to demonstrate the validity of an algorithm or is it also necessary to be part of a structured theory? We believe that it is enough to ask whether an algorithm is valid to have already fully entered the domain of mathematics. Therefore, we want to underline that "proof" is not a neutral concept, it is not objective and it is not a point of arrival: it is the beginning of a construction.

[10]Chemla, 2012. p. 61.

1.3 Medieval cosmological vision

This section is based on the studies by Paolo Rossi (Rossi, 1968) and Lino Conti (Conti, 1992).

Let us start to delve into what will be the specific theme of our work, the mathematical proof in physics. To understand the importance of Galileo in the history of physics, it is appropriate to have an idea of the millennial system that he contributes decisively to destroy.

The medieval universe is not continuous and his physical laws are not the same in all places; There is a separation between the *celestial world* and the *sublunary world*.

In the sublunary world, everything is a mixture of four elements: earth, water, air, fire.

The weight of a body is given by the proportion of the four elements in his composition. In fact, earth and water naturally tend downwards, air and fire upwards. From the restlessness of the four elements follows the continuous change that characterizes the sublunar world. Without the mixing of the elements everything would be at rest: we would have in the center a sphere of earth, surrounded by one of water, surrounded by one of air, surrounded by one of fire, all motionless.

Upward or downward motions are absolute, not relative. Therefore the bodies and the preponderant elements from which they are composed, tend to reach their natural place, following a *natural motion*. this is the case of a body in free fall, of the flame going upwards as well as the air bubbles in the water. The motions that counteract the natural motions, i.e. when the body is subjected to a force contrary to its nature, are called *violent motions*. Whoever throws a stone commits this kind of violence, then when the force acting on the stone is consumed, the stone can return serenely to its natural place.

In the celestial world, things are different. The motions are all regular, non-violent, everything is immutable and eternal. The celestial bodies are not composed of the same elements, but of a fifth element, or quintessence. The planets and the sun are actually fixed, but rest on concentric celestial spheres, always made of quintessence, and these speres are rotating. In the celestial world the only type of motion is the circular one, which is perfect because it has no beginning or end. On the other hand, in the sublunary world we also have linear motion, limited in time. The universe is limitated and the Earth, which cannot rotate due to its lower nature, is immobile in the center of the universe.
As Paolo Rossi points out his conception is not an invention of Aristotle but is the transposition into the physical world of a purely geometric and abstract model of Eudoxus of Cnidus, from the first half of the fourth century BC. The Aristotelian theory was then re-elaborated by Ptolemy of Alexandria, in the 2nd century AD. His works are the foundation of astronomical knowledge throughout the Middle Ages. Ptolemaic theories seem able to explain and predict celestial motions. The observation contrasts with a model in which the planets rotate while remaining

fixed. In fact their distance seems to change, they seem to stop, go back, change speed, but all this must be explained in terms of circular motions. Ptolemy implements mathematical theories: the theories of eccentrics, epicycles and equants (the first two preceding Ptolemy) that perform this task. After that, the picture of the universe that persists in European culture is a complex mixture of Aristotelian physics and Ptolemaic astronomy, with further contamination due to the Fathers of the Church and the philosophers of the Scholasticism. If the picture is articulated, however, we can identify six points[11] in particular that modern science will have to clash to assert itself.

1. The space-time discontinuity between the celestial and terrestrial worlds. The division of the universe into two parts: one perfect, the other imperfect and subject to becoming.

2. The exclusively circular conception of celestial motions.

3. The postulate of the immobile Earth at the center of the universe, became a "truth of common sense", which seem to have confirmation in the Holy Scriptures.

4. The belief in the finitude of the universe and in a closed world, a belief linked to the doctrine of natural places.

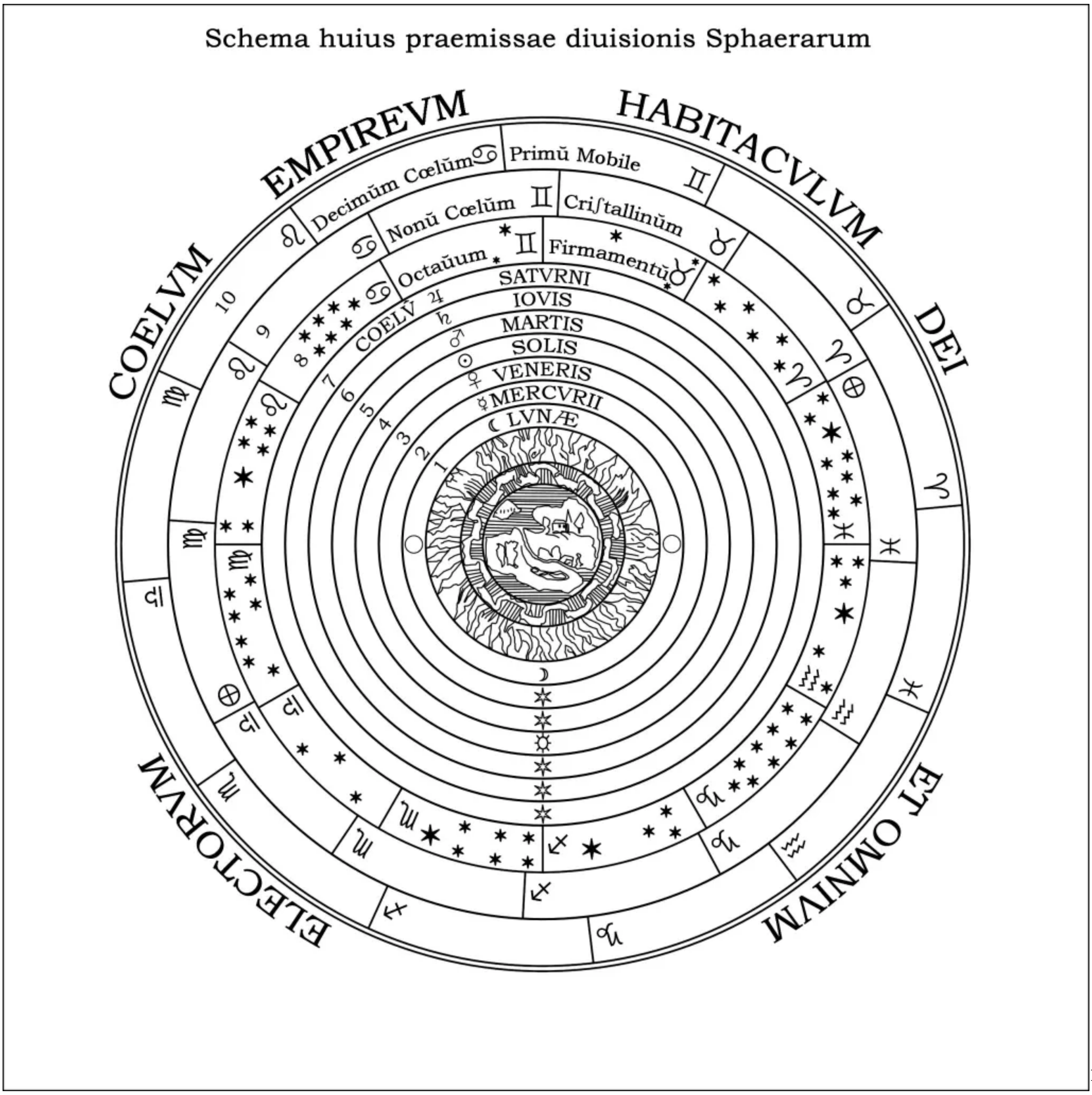
5. The belief, linked to the distinction between natural and violent motions, that every movement is explained as dependent on the nature of the body or caused by an engine that produces it and keeps it in motion during the movement. This point in particular is crucial in the perspective of our work.

6. The division between the hypotheses of physics and those of astronomy, as if the sphere of theoretical speculation were separate from that of mathematics. This point is precisely the general subject of our work.

Let us take up the fifth point, which concerns the conception of motion. If the engine must be in contact with the moving body, any remote action is inconceivable. The problem becomes complicated in the case of the (violent) motion of the projectiles. An arrow should fall to the ground as it detaches from the bowstring. One first explanation, which appears weak to modern eyes, is that the compressed air in front of projectiles immediately occupied the void that tended to form behind them, somehow pushing the projectile. A second explanation is the medieval theory of *impetus*, a theory that is more interesting and closer to common sense.[12] It is a force possessed by the object that is progressively consumed during motion. Aristotelian astronomy and theory of motion, areas which, as we have seen, are connected, are those in which Galileo's work will impose an important discontinuity.

[11] Rossi, 1968.

[12] Hestenes, Wells, Swakhamer, 1992. *Force Concept Inventory* is a test measuring mastery of concepts commonly taught in a first semester of physics at University.



[13]

[13]Peter Apian's Cosmographia, 1539. Picture from: https://classicalastrologer.com/nature-of-signs-planets-in-classical-astrology-2/

1.4 Mathematics meets science

In the 17th century Descartes and Galileo brought scientific practices to a revolution, choosing the concepts that science should use, what are the goals of the investigations and the methodology to be used.

Descartes in 1637 published Discourse on the Method of Rightly Conducting One's Reason and of Seeking Truth in the Sciences, also known more simply as the Discourse on the Method. It is a work that appears halfway between science and philosophy and may seem to be neither one thing nor the other. It is appropriate here to make a clarification, in the words of the philosopher Carlo Sini:

> S'intende che al suo tempo, come ancora al tempo di Newton e Leibniz, non era accaduta quella separazione fra scienza e filosofia [...] Al tempo di Cartesio la parola "filosofia" ricopriva ancora un ambito di problemi e di ricerche che andavano dalla metafisica alla logica, alla morale e alle scienze naturali; e sebbene Cartesio, come già Bacone, Galileo o Gassendi, operasse vigorosamente per affermare l'autonomia di metodo e di contenuto delle scienze naturali, di fatto svincolandole dalla filosofia scolastica e in particolare dall'aristotelismo, nondimeno era ancora sua viva preoccupazione quella ricerca dei principi primi e dei fondamenti ultimi e indubitabili che erano propri della filosofia. [14].

We can specify that Descartes not only deals with first principles (Galileo and Newton do it equally) but he does it in a philosophical, speculative way. For this reason, although it comes very close, we cannot strictly speak of scientific method in Descartes. Nevertheless Descartes is convinced that reality should be observed in a mathematical key. He says to:

> do not admit or hope for any principle for physics other than those of geometry and abstract mathematics, because thus all phenomena of nature are explained and can be proved. [15]

And also, without frills:

> give me extension and motion and I will build the universe [16]

It is a mechanistic philosophy: if extension and motion are mathematically expressible, all the phenomena, including life, can be described mechanically. Descartes

[14] "In his time, as still in the time of Newton and Liebniz, that separation between science and philosophy had not yet occurred [...] At the time of Descarte the word "philosophy" still covered a field of problems and research ranging from metaphysics to logic, morality and natural sciences; and although Descartes, like Bacon, Galielo or Gassendi, worked vigorously to affirm the autonomy of method and content of the natural sciences, de facto freeing them from scholastic philosophy, and in particular from Aristotelianism, nevertheless that search for first and for the ultimate and indubitable foundations which were proper to philosophy, was still crucial." Carlo Sini, *Introduction* to Descartes, 1993, p. XII.

[15] Kline 1991, p. 380.

[16] Ivi, p. 381.

constructed a philosophical system that disrupts the influence of scholasticism, attributing the causes of natural phenomena to purely physical events. His works had a strong influence, his deductive and systematic philosophy also impressed Newton.

As Kline poits out, much of Galileo 's philosophy coincides with that of Descartes, but it is Galileo who formulates concrete procedures for modern science, demonstrating their effectiveness with his work. His method, which is still considered the scientific method, is exposed in his work Discourses and Mathematical Demonstrations Relating to Two New Sciences. The two sciences mentioned in the title concerned the strength of materials and the motion of objects. It is a text secretly transported to Holland and published there in 1638, when the Inquisition had already prohibited its publications.
Galileo is an eclectic scientist: acute astronomical observer, inventor of the microscope and the pendulum clock, he also successfully deals with the theory of sound waves.
As Descartes, he broke with mystical speculative theories in favor of a mathematical and mechanical view of nature, but he deviated from him with respect to the individuation of the first principles. Principles cannot be identified by intellectual speculation, as a Pythagorean would have believed and as then Aristotle had definitively established. The relationship between principles and experience is however delicate and we will focus on this in the second part of this section.

With regard to experimentation, as it will be for Newton fifty years later, the figure of Galileo is a transitional figure. He was convinced that few crucial experiments would provide the principles. Many of the experiments, that he later conducts, were thought experiments.
Once the principles have been established, most of the work is of a mathematical-deductive type, for both Galileo and Newton. Galileo is satisfied with the theorems that follow from a single principle as with the discovery of the principle. The mathematical approach is also perceptible in the abstraction process with which Galileo models physical phenomena. In the study of motion it eliminates air resistance and friction, just as mathematicians eliminate the molecular structure, color and thickness of lines to study their fundamental properties. Galileo take in consideration motion in a vacuum even if this contradicts Aristotle and even Descartes. The method of abstraction deviates from reality, but leads back to reality with greater force than not considering all the factors.

Another important point to underline is that Galileo wants to find quantitative axioms. For Aristotle a ball falls towards the Earth because every object seeks its natural place and the natural place of heavy bodies is the center of the Earth. A quantitative statement says that the speed of the falling ball is:

$$v = 9,8t$$

Where t is the fall time. To our eyes it is probably pleasant to recognize a clear equation in the midst of so many philosophical dissertations, even if this equation gives no explanation the reason why the ball falls. That is what equations do:

they do not explain but they describe. Aristotelians spoke of essences, of natural places, of violent motion and natural motion, the concepts that Galileo chose the property of being measurable, i.e. with values that can be related by equations.

Similarly, Newton had good reasons for preferring quantitative mathematical laws over physical explanations. The central physical concept of its celestial mechanics, the force of gravitation, had no explanation. This was the passage to modern science, the possibility of a mathematical description even where physical understanding was lacking. We conclude this brief introduction to scientific thought with the words of Morris Kline, who describes the virtuous circle that has arisen between mathematics and physics in this way:

> Since science had come to depend heavily on mathematics, almost being subordinated to it, scientists extended the domain and techniques of mathematics, and the multiplicity of problems provided by science gave mathematicians numerous and important directions of development for their creative activity. [17]

So far everything seems wonderful and perfectly harmonious. However, one might wonder if with the advent of the scientific method the sphere of knowledge was suddenly freed from the ancient metaphysical legacies and from the authoritative weight of Greek philosophers. Unfortunately it is not that simple, and it is the theme we will deal with in the next section, in which we will consider some aspects of Galileo and Newton's thought. We will see that tradition and modernity not only continue to coexist, but are intertwined and difficult to separate.

[17] Kline 1991, p. 392.

1

1.5 Concepts of proof in Galileo and Newton

The scientific revolution, which has as symbol Galileo and finds an ideal fulfillment in Newton's Principia, is not a simple and straightforward process. It would be misleading and erroneous to think of summarizing it as a mere passage from abstract philosophy to mathematical empiricism and it is also not correct to think of it as a radical break with the Aristotelian tradition. We have already mentioned how Copernicus thought himself in the wake of the Aristotelian tradition. But this could be said of most of the scholars formed in the sixteenth century, with a scholastic training.

As Ennio De Bellis[18] points out, the reference teacher of Galileo's first logical thought is certainly Paul Valla, and reference texts of Valla's logic course are part, like those of other European universities, of a rigid Aristotelian curriculum based essentially on *Categoriae*, *De interpretation*, *Analytica priora*, *Analytica posteriora*, *Topica De sophisticis elenchis* by Aristotle, *Isagoge* by Porfirio and *Summulae logicales* by Pietro Ispano.
How profound is the influence of Aristotle's thought on the Pisan physicist? Let us consider Galileo's *Tractatio de praecognitionibus et praecognitis* in William Wallace's translation. To the question "Should the principles of sciences be so evident tha they cannot be proved by any reasoning?" Galileo replies:

> It seems so, first, from Aristotle, text 15, [a] because principles are supposed, not proved; again [b], because it is the task *of the metaphysician* [19] to proove principles, therefore not that of the other sciences; again [c] because principles must be foreknown prior to any demonstration, therefore they are not proved. [20]

Reading this passage we could almost think of a metaphysician who dictates a priori axioms that the scientist then uses. But that is not at all the case. Shortly after Galileo continues:

> Certain moderns distinguish two kinds of principles: some are of the object or in the order of being, others are of knowledge or in the order of knowing. They teach that principles in the order of being can be demonstrated *a priori* from principle in the order of knowing. But quite the contrary: for principles in the order of being can be demonstrated only a {*a posteriori*} from principles in the order of knowing. Averroes, in the first book of the Posterior Analytics, com. 22, and in the seventh book of the *Metaphysics* whom practically everyone follows, states that principle in the order of being, when unknown, can be demonstrated *a posteriori*.[21]

[18]De Bellis, 2016.
[19]My italics.
[20]Galileo, Logical treatises, p.90.
[21]Ibidem.

And therefore while fully inscribing the Scholastic tradition, and seeking from this legitimacy, at least the boundary that separates a physical law from a principle (in the order of being that can be demonstrated through principles in the order of knowledge) seems to be thinning. Even if in theory we still have two separate sciences, metaphysics which defines principles and physics which studies phenomena, at least in Galileo's early stage. But what is more precisely the relationship between principles and experience?
In this regard, Ludovico Geymonat explains that Galileo is well aware that the axioms and general definitions will not, except in exceptional cases, be derived from experience, such as for example the definition of naturally accelerated motion, which uses concepts such as that of instantaneous speed, not relevant to experience.

> Even in this case, however, the theory based on axioms so far from experience could - according to Galileo - turn out to be an authentic scientific theory, provided that it satisfies the condition that the consequences rigorously deduced from the aforementioned principles are confirmed in experience. In other words: it is not necessary that all the propositions of the theory be adherent to the facts; instead it is necessary that all the facts of the field of phenomena studied can be inserted into the theory.[22]

In this sense Galileo separates physical theory and pure mathematics. Mathematics does not ask for any control from experience, continuing to hold true regardless of whether the figures studied exist in reality or not. Physical theory, on the other hand, proposes to arrive at phenomena, and if its consequences are not confirmed by these, it no longer has scientific value. As we know, consistently with his studies and its methodological approach, Galileo openly rejected the Ptolemaic system, with all that follows. We could therefore conclude that Galileo is respectful of tradition from a formal rather than a substantive point of view, and that his thinking is in fact already modern.

Let us now consider Newton's position with respect to this order of questions. First of all, in the eighteenth century, an Aristotelian terminology is still in use. In particular Newton uses, while reinterpreting it, an important Aristotelian concept: that of efficient cause.
Newton's view is of theological type. First of all, the *Principia* consider what he calls *final causes*. In fact, the motion of celestial bodies acts according to the law of universal gravitation, but their regular position can only be explained with

> The design and dominion of an intelligent and powerful being.[23]

Secondly, the *Principia* also provide *formal causes*, namely the laws of motion that determine the trajectories that bodies can follow. Final and formal causes have a different ontological status.

[22]Geymonat, 1970, vol II, pp. 205-206.
[23]Newton, Principia, p. 941.

As Steffen Ducheyne and Erik Weber[24] point out, the main concepts of the Principia have a certain ambiguity: force, attraction and gravity may seem purely mathematical or may seem causal. Some passages could lead to a positivistic reading of the principles, others refer to the importance that Newton attributes to causality. This apparent ambiguity could be justified by the fact that Newton's theory acts on two very distinct levels: the plane that describes how things work, in a mathematical way, and that which tells us why they happen, from a theological point of view. Let us consider the following comment to *Definition VIII*:

> Porro attractiones et impulsus eodem sensu acceleratrices et motrices nomino. Voces autem Attractionis, Impulsus, vel Propensionis cujuscunque in centrum, indifferenter et pro se mutuò promiscuè usurpo; has vires *non Physicè, sed Mathematicè tantùm considerando*. (italics mine) Unde caveat lector, ne per hujusmodi voces cogitet me speciem vel modum actionis causamve aut rationem Physicam alicubi definire, vel centris (quae sunt puncta Mathematica) vires verè & Physicè tribuere; si forte aut centra trahere, aut vires centrorum esse dixero. [25]

But in Newton's view, bodies are passive and moved by active principles. The celestial motions do not have origin in mechanical causes. Therefore the fact that he does not consider physical causes and the sites of forces does not imply a rejection of the efficient cause in general. Ducheyne remembers how in the scholium of section 11 of the book Newton defines the steps of the physical investigation of gravity thus:

> In mathesi investigandae sunt virium quantitates et rationes illae, que ex conditionibus quibuscunque positis consequentur: deinde, ubi in physicam descenditur, conferendae sunt hae rationes cum phaenomenis; ut innotescat quaenam virium conditiones singulis corporum attractivorum generibus competant. Et tum demum de virium speciebus, causis et rationibus physicis tutius disputare licebit. Videamus igitur quibus viribus corpora sphaerica, ex particulis modo jam exposito attractivis constantia, debeant in se mutuò agere; et quales motus inde consequantur.[26]

[24]Ducheyne, 2007, p. 266.

[25]"I use interchangeable and indiscriminately words signifying attraction, impulse, or any sort of propensity toward a center, considering these forces not from a physical but only from a mathematical point of view. Therefore let the reader beware of thinking that by words of this kind I am anywhere defining a species or mode of action of a physical cause or reason, or that I am attributing forces in a true and physical sense to centers (which are mathematical points) if I happen to say that centers attract or that centers have forces." Newton, Principia, Definitio VIII.

[26]"Mathematics requires an investigation of those quantities of forces and their proportions that follow from any conditions that may be supposed. Then, coming down to physics, these proportions must be compared with the phenomena, so that it may be found out which conditions (or laws) of forces apply to each kind of attracting bodies. And then, finally, it will be possible to argue more securely concerning the physical species, physical causes and physical proportions of these forces. Let us see, therefore, what the forces are by which spherical bodies, consisting of particles that attract in the way already set forth, must act upon one another, and what sorts of motions results from such forces." Newton, Principia, Scholium Liber Primus, Sectio XI.

Therefore physical agents are inferred from the mathematical properties present in nature. But as far as gravity is concerned, Newton has established that a non-material cause must be introduced, he does not conceive the idea of action at a distance, and the first cause of celestial motion is God itself. Hence Newton, the inventor of calculus, in competition with Leibniz, the writer of a Euclidean setting system has God as a first principle. Not as an a priori axiom, but a posteriori, through a method that is not substantially different from that of Galileo. The conclusion, however, is stronger and more difficult to reconcile with modern scientific sensitivity. As Geymonat points out[27], the results pursued by Newton were mainly two: on the one hand to provide a new proof of the validity of religion, leaning it on the results of science, and at the same time to give indirect confirmation to science. This marriage between science and religion, however, did not last long, if not in Freemasonry. For the church, Newton's theology is too heterodox and became a point of friction between a "rationally based" religion and a religion based on the Gospel.

With regard to the initial question, we cannot say that Newton has completely emancipated himself from metaphysics. What is often identified with the birth of modern physics may not seem entirely modern to us. After all, every revolution is relative to the context in which it takes place. It may not be obvious (perhaps to some, yes) that the invention of the differential calculus was born in conjunction with an innovative theological conception. In any case it is so, and with Newton not only mathematics but modern mathematics, largely invented by himself, has become part of the scientific discourse.

[27] Geymonat, 1970, pp. 647-648.

Chapter 2

Roles of mathematics in physics education

2.1 Mathematics as a language, a possible role of math in physics education

In the first chapter we have made a historical and conceptual overview, necessarily biased and far from any kind of exhaustiveness, to give at least an idea of the role of proof in science. Our goal is now to see what the roles of proof can be in physics education. This type of discourse can only be interdisciplinary. In fact, the concept of demonstration from Euclid onwards it is what characterizes mathematics and philosophy as disciplines, even before characterizing science. As we will see, mathematics and physics, while maintaining a specific identity, are often inseparable fields. So, assuming we have given an idea of what a mathematical proof is and why it is important in the scientific field, we want try to understand how it can be used in physics education.

Let's start by taking a step back and asking ourselves: what are the roles of mathematics in the study of physics? The first answer that can come to mind is probably: mathematics serves physics as an indispensable tool for making accounts and providing predictions. This fact is undeniable and it is a great start, however, from a didactic point of view, there can be problems when this fact becomes an exclusive perception. For students (henceforth by students we mean high school students, although the problems we present may obviously also concern university students) calculation is often perceived as the only function of mathematics in physics.

In contemporary literature, as we shall see, ample space is given to the conception that mathematics, in itself but in particular within the didactics of physics, is instead a language.[1] And a language, unlike a mere tool, is something that is functional not only to practice, but also to the development of ideas.

The conception of mathematics as a language and the link between this language and the study of physics is not entirely new. We will not deal here with the

[1] In this section in particular we will refer to the work of Pietrocola, 2010.

declinations of this concept in the Platonic and Pythagorean doctrines, but we will refer to a very famous quotation from Galileo. In turn, this quote lends itself to many interpretations.[2] We limit ourselves, for now, to taking it as a symptom of how much this link between mathematics and physics, which develops in terms of a language, is something that comes from afar, from the roots of scientific thought.

> La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.[3]

As mentioned in this chapter we want to deal the interdisciplinarity between mathematics and physics, and from this point of view we can ask ourselves what the passage from Galileo can suggest. Well, the physical world is not something whose essence can simply be revealed. It is written in a language that is not a natural language, it is not what we commonly use to communicate. This means that any form of knowledge can only be reached through a translation operation. But our modern sensibility knows well that every form of translation is both an act of love but also of betrayal. Systematically something is lost, in every translation, and more translations are just as legitimate. There is not only the intrinsic logic of the phenomenon we want to describe, but also the logic of the grammar we use to express ourselves, which we can never ignore. How can this awareness be usefully transferred into teaching?
We will see various aspects and various responses of specialist reading, but one in its naive simplicity is particularly important to us: to make students aware of the path they are taking and the tools they are using.

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[2] Often a Platonist interpretation is given to this passage: mathematics would be the way to grasp the "true" essence of things, beyond the appearances offered by phenomena. But a different interpretation is possible: only mathematics gives us a sufficiently powerful and precise tool to understand the universe. To understand where the metaphor of the book and of the characters comes from, it is necessary not to isolate the passage from what precedes it:

It seems to me, besides this, that I perceive in Sarsi a firm belief that in philosophizing it is necessary to rely on the opinions of some famous author, so that our mind, when not married to another's speech, would have to remain in all sterile or infertile; and perhaps he believes that philosophy is a book and a man's fantasy, like the Iliad and Orlando Furioso, books in which the least important thing is that what is written there is true. Mr. Sarsi, this is not the case. The philosophy . . .

We see that the emphasis is actually on the fact that philosophy (understood naturally as "natural philosophy," or "science" in today's language) is not a man's fantasy, nor is it based on authority ', but it is written in the book of the universe. Mathematics is the language you need to know to decipher it". Fabri, 2010.

[3] "Philosophy is written in this grand book, which stands continually open before our eyes (I say the 'Universe'), but can not be understood without first learning to comprehend the language and know the characters as it is written. It is written in mathematical language, and its characters are triangles, circles and other geometric figures, without which it is impossible to humanly understand a word; without these one is wandering in a dark labyrinth". Galilei, Saggiatore.

Furthermore, even if we will not deal with it specifically, it seems fair to point out that the treatment of mathematics as a language is, as one might imagine, also useful in the teaching of mathematics. In reality, the question is anything but trivial, and rather delicate, we just make a vague mention of it. Most people probably have an idea of what the study of physics is, and if this idea fits into something like "the study of the laws of nature", however crude, it is not that far from the truth. On the other hand, those without a scientific background, including many students, think often that mathematics is "the study of numbers" or "how to calculate". Well ... in this case we are much further away from something that resembles the specificity of the discipline. What kind of knowledge is mathematical knowledge? Bertrand Russell explains this with his brilliant concise clarity:

> Mathematical knowledge, it is true, is not obtained by induction from experience; our reason for believing that 2 and 2 are 4 is not that we have so often found, by observation, that one couple and another couple together make a quartet. In this sense, mathematical knowledge is still not empirical. But it is also not a priori knowledge about the world. It is, in fact, merely verbal knowledge. "3" means "2 + 1," and "4" means "3 + 1." Hence it follows (though the proof is long) that "4" means the same as "2 + 2." Thus mathematical knowledge ceases to be mysterious. It is all of the same nature as the "great truth" that there are three feet in a yard.[4]

We are not suggesting to offer a speech in these terms to high school students, but it might be helpful if it was part of the teacher's cultural background. Mathematics could be presented not as revealed knowledge, as an eternal world of ideas, but as a construction, which should be historicized, and as a discourse. At least this is the point of view we have chosen to take as an interpretative paradigm for our work.

But scientific language is not made up of mathematics alone, it is an hybrid language, incorporating aspects of common language and formalized language. As Pietrocola points out, an excessive emphasis on formal language, which represents the final stage of the scientific process, can be counterproductive on students, if an appropriate interpretation is not provided. Students may be led to think that scientific work is a mere discovery of pre-existing facts, without understanding that scientific work is at least in part, but a decisive part, an invention [5].

In a perfect world, the teacher has the task of pointing out how mathematics is a language and also giving instructions for the correct use of this language.

> There is a myth about the relationship between physics teaching and mathematics teaching that can be overturned when there is clarity concerning the differences between both of these skills – while the first skill can be obtained outside physics education, meaning, in subjects

[4]Russell, 1945, Chapter XXXI.

[5]This is why we believe that even in the field of didactics the assumption of a Platonist point of view is less constructive than emphasizing the construction and convention aspects of mathematics.

> exclusively mathematics; the second one cannot be. The capacity of dealing with mathematics within its own situations does not warrant the capacity of using it in other areas of knowledge, as physics. In other words, to have technical command of mathematics does not guarantee the capacity of employing it to structure thought in other domains. This implies that there must be a didactic-pedagogical intention in preparing the physics students to make structural use of mathematics.

The fact that technical skills are not converted into structural skills generates what in the literature is called an epistemological obstacle. In the third chapter of this thesis, in particular, we will see concrete examples in which we try to overcome obstacles of this type and create structural skills.

2.2 Mathematization of physical phenomena

If the proof made with pen and paper constitutes an iconic moment of truth in mathematics, the equivalent in physics is given by the experiment in the laboratory. So we may be led to think scientific experiment can have a function similar to that of a proof: to convince of the validity of a physical law, just as a proof convinces of the validity of a mathematical conjecture. There is an obvious important difference: a proof in mathematics, logically correct and whose hypotheses have all been made explicit, is sufficient, while no physical experiment is conclusive in itself.

In physics, as in mathematics, there are theorems and their corollaries, however, they are not the consequence of arbitrarily fixed axioms, but the consequences of physical laws considered valid. Physical laws cannot be verified in the sense that this expression has in mathematics. They would need to pass an infinite number of experimental tests. There is therefore no "physical proof", experimental verification always has a different meaning than the proof of a theorem. Experimental verification is an empirical stage of the scientific method, the proof of a theorem is an eternal truth. The price to pay for "eternity" is to have chosen the axioms in a completely arbitrary way. As far as physical laws are concerned, it is always possible that a new experiment necessitates a new reformulation.. Physics is therefore the slave of two severe masters, on the one hand mathematics, with its formalisms and the hypothetical deductive method, on the other, experimental verification, with its merciless confrontation with a reality that is not always easy to approximate. In this section we want to focus on how mathematics is a cumbersome guest in the world of physics, and in the didactic consequences that this entails.

> For many physicists it is impossible to consider a physical theory without mathematics. Instead of seeing mathematics as an external useful tool, it is rather considered to be the of nature (or even of God according to James Jeans and Paul Dirac). In this sense it penetrates into the physicists' discourse, since theoretical explanations in physics are frequently enabled by the deductive character of the mathematical formalism. It also helps to structure physical thought, as mathematics often provides analogies that enable physicists to think about unknown phenomena and serves as a reasoning guide in the path to abstraction.[6]

In Uhden's article just quoted, reference is made to a text by Gingras: *What did mathematics do to physics?*[7], Where a historical perspective is adopted to argue that "physics is mathematical in its formulation". Indeed, it is difficult to establish whether the leading scientists of the seventeenth and eighteenth centuries, including Newton, Euler, Lagrange, Fourier, the Bernoullis were more mathematicians or physicists.

Gingras gives us a classification of the main three effects of the mathematization of physics :

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[6]Uhden, 2012.

[7]Gingras, 2001, pp. 383-416.

1. *Social*: The sitematic use of mathematics makes the study of physics impractical for laymen. We observe that a consequence of this is the extreme difficulty encountered in the creation of popular science texts, where the author tries as much as possible to "give an idea" without distorting what is being explained.

2. *Epistemological*: The mathematization of physics has changed the meaning of the term explanation. The explanation of a physical phenomenon has been gradually replaced by the need to represent it with mathematical formulations, hence a transition to formal language.

3. *Ontological*: The mathematization led to not considering as existing substances such as caloric, cartesian vortices, electric fluids, the luminiferous ether or the caloric. We have a world described by mathematical relationships and not by essences that have certain properties.

With respect to the epistemological aspect, we can still observe how historically we start from an intuitive physical principle and mathematization is a subsequent step, which often involves a redefinition of the principle. The example that we will face in the third chapter, regarding the parabolic motion studied by Galileo, focuses precisely on this.
The mathematization of physics has significant practical aspects in the development of the discipline. Just as mathematics is for building other mathematics, mathematical physics is for building other physics.

> The mathematization of a physical theory makes it not only a more concise and precise representation, but it also serves as a reasoning guide for thinking about new phenomena. Formal analogies used to think about electricity as a fluid, light as a wave and an electric circuit as a spring-mass system are examples of the reasoning by formal similarities in physics. According to Feynman, "the equations for many different physical situations have exactly the same appearance. This means that having studied one subject, we immediately have a great deal of direct and precise knowledge about the solutions of the equations of another."[8]

This is what Wigner defined as "unreasonable effectiveness"[9], the fact that more descriptions of reality are obtained from a physical equation than they were initially thought, even in very different contexts. While noting that all this is undoubtedly fascinating, we want to consider what the problems may be at the didactic level.

It is often taken for granted, and to some extent it is indisputably true, that mathematical prerequisites are necessary to take physics courses. Mathematical skills can be conceived as tools that the student should have in an imaginary toolbox, always ready for use. The problem is that a purely algorithmic learning of

[8] Uhden, 2012.
[9] Wigner, 1960, pp. 1–14.

mathematical procedures does not in any way guarantee the subsequent understanding of physical phenomena. There are areas in which the distinction between mathematical and physical concept is rather artificial, as an understanding from a mathematical point of view is essential for the understanding of the physical concept, but also understanding from a physical point of view helps the understanding of the mathematical concept. The perfect example is given by the concepts of velocity and acceleration. The concept of derivative of a function was clearer to me, or at least I had a useful representation of it, when I discovered that velocity is obtained as a derivative of displacement and acceleration as a derivative of velocity with respect to time. The physical concept of acceleration is often difficult to understand for high school students because it is a rate of a rate: acceleration is the time rate of change in velocity, which also is a rate, namely the time rate of change in position. Of course it is not possible to talk about derivatives when the concepts of speed and acceleration are introduced for the first time at school, at most they can be revisited at a later time when the necessary calculus tools have been introduced.

Uhden gives a little example of a didactic approach to a process of mathematizing a physical phenomenon related to these concepts, a free fall problem. He is asked to describe the movement of a body falling from a certain height due to gravity. The goal is to determine the position of the body as a function of time, $s = s(t)$.

First of all it should be noted that the problem is idealized: body size and air resistance are not considered. The problem is also mathematized: time and space are represented by real numbers. This means that we are already within a precise physical-mathematical model. The didactic approach consists in gradually increasing the level of mathematization.

We point out first of all that the relationship we are considering is not linear. The more time passes, the more the body travels greater distances for the same amount of time. That the increase in velocity is $10m/s^2$ could be discovered through an experiment or a simulation.

We proceed by giving some representations of the speed increase: first a table that reports different values of the speed v at a time t and then the relative graph(v vs t) from which the relation $v = g \cdot t$ can be deduced. This is already a higher level of mathematization, followed by the conclusive one: interpreting the area under the graph as the distance $s(t)$ traveled by the body.

To arrive at this conclusion we can use the graph of uniform motion as a bridge. In this case the area has the shape of a rectangle whose sides correspond to v and to t. Since here velocity times time equals the displacement $s(t)$, the area can be interpreted as the displacement of the body.

To pass from the case of uniform motion to that of uniformly accelerated motion, we can consider the triangle that has area $\frac{1}{2}v \cdot t$. Since $v = g \cdot t$ we have:

$$s(t) = \frac{1}{2} \cdot v \cdot t = \frac{1}{2}(g \cdot t) \cdot t \Rightarrow s(t) = \frac{1}{2}gt^2$$

Nothing particularly original and complicated, but we want to underline how the key point is to provide students with an awareness of the tools and method that is being used.

2.3 Lack of physical reasoning in classroom

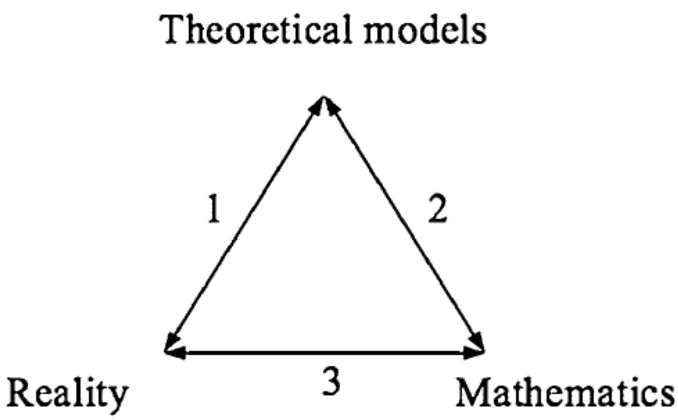
Another crucial aspect that mathematics plays in the study of physics, in research as well as in teaching[10], is the construction of models. Models are a tool that scientists use to create new theories, test ideas, analyze data, as well as teachers use them to make students understand what they are talking about. Making it clear that scientific progress necessarily goes through the development of models is a way to make students understand how science works.
Important examples are those for weather forecasts and those for the analysis of atomic structures. In general, the model creates an approximation of physical reality in accordance with a theoretical description.
Still everything looks beautiful and harmonious in theory, but we will see that there is a risk that this is lost due to a "hyper-mathematization" of physics. By "hyper-mathematization" we mean something stupid like: "to solve the exercise find which are the correct variables to insert in the appropriate formula".

We will consider then a study of 2015 by Hansson, Juter, and Redofors [11]. The study aims to highlight the dynamics that are created between reality, theoretical model and mathematics, during physics lessons. Lessons held in an upper-secondary school by the same teacher in three classes were taken into consideration, for a total of seven lessons. The idea stems from the belief that the complex relationship between a theoretical model and the real world can be one of the causes why physics is seen by students as too difficult and at the same time not very interesting. The awareness or not that students have of the relationship between observations, theoretical models and reality inevitably influences the ability to apply their knowledge to real world situations [12].
Various studies, including those we have already mentioned by Uhden et al. and by Pietrocola, highlight how the ability to transfer mathematical knowledge to an applicative field is by no means automatic. In this case we talked about the difference between the "technical" and the "structural" role that mathematics has in physics instruction.
In fact, in the upper-secondary school the study of physics is often reduced to solving standard problems, those that are generally found at the end of each chapter in the textbook. Teachers often assume that a good understanding of physical concepts and patterns is behind the ability to perform these exercises. Unfortunately, this is not the case at all, as the very interesting *Force Concept Inventory* by Hestence, Wells and Swackhamer shows [13]. It is about a catalog of the main misconceptions relating to Newton's laws, starting from a questionnaire based on qualitative questions, tested on a sample of 1500 high school students and 500 university students. The funny and dramatic data that emerge suggest precisely that there is not necessarily a correlation between "algorithmic ability" in carrying out the exercises and understanding of the principles. The interesting aspect of the *Force Concept Inventory* is that it does not simply show an absence of physical

[10] Schwarz, 2003.
[11] Hansson, 2015.
[12] Hansson, 1958.
[13] Hestence, 1992.

principles, but the replacement of Newtonian principles with alternative, unscientific, common sense principles. Among the most common we find the principle of "impetus", which recalls the Aristotelian principle, according to which a moving object has a certain energy. We then have the principle of "active force", for which a motion cannot be given without a cause, for which the concepts that speed is proportional to the force and that the acceleration is proportional to the force are equivalent.
The Force Concept Inventory, however, limits itself to detecting misconceptions without identifying a specific cause. We can hypothesize that upstream there was not enough work in clarifying the relationship between reality and modelling, assuming that once the principles were exposed and practiced with typical exercises, everything was fine.

This is the framework in which the study by Hansson et al. fits. Here reality, theoretical models and mathematics are considered as distinct entities, although of course not separate. Reality simply refers to objects or phenomena observed in the real world. Theoretical models can already be mathematical or formulated in a qualitative way. By mathematics we mean concepts, theorems, representations and the usual mathematical methods.



The classes of students considered were as follows: 30 first year students, studying mechanics, 10 third year students, studying optics and atomic physics and another class of 7 third year students, studying electric fields. Both the teacher and the students were filmed by multiple cameras at the same time. classes were observed during lectures, group exercises and worklabs. Here we limit ourselves to quoting some fragments of a lesson on electric fields, by way of example. The lesson begins with the teacher demonstrating the strength of electric fields. The students could see the voltage required for a spark in a parallel-plate capacitor charged by a hand-powered generator.

> The teacher asked the students why there is a spark and concluded that the capacitor wants to equalize the charge (link 1). The spark oc-

> curred more often and with less required voltage when the distance between the plates decreases (link 3), "Is this logical?" asked the teacher, thus directing attention to relationships between Reality and Theoretical model (link 1), and the students thought so. The teacher asked whether they in their previous physics course ever calculated the field strength required for a spark. The students confirmed that they had done so and gave the formula E = U/d which the teacher wrote down on the whiteboard. [...] Then, the teacher drew a parallel-plate capacitor on the whiteboard (horizontal, with negative plate up and positive down) and asked "what would happen if you put an electron in the electric field", noticing that an electron would accelerate downward as it travels horizontally between the plates. The students were uncertain, and the teacher pointed out that there is no force acting upwards and explained that there is a net force, which willmake the electron accelerate. The teacher's and students' reasoning was now instead focused on the relation between Reality and Theoretical models (link 1) with an emphasis on Theoretical models. The teacher draws attention to that they now can use the formulae $F = ma$ and ($E = F/Q$ and write) $F = EQ$, noticing, "F is the force that the electron senses in the electric field." The focus was now on formulae manipulation (link 2, technical), and in this context, one student commented on Coulomb's law asking "did it not also contains Q?". The teacher replies that "in that case it's about the force between two charges". The student browsing his formula book continues "what about $U = E/Q$" and the teacher replies, *"In that case E does not mean field strength but energy, it's a bit tricky"*. (italics mine[14]) Then, the teacher, for the largest part of the lecture, solved problems related to field strength on the whiteboard. The focus was on manipulation of formulae and finding formulae that can be used to solve the problems (link 2, technical). In this, the teacher gave examples very similar to the textbook problems, which the students were going to work with later on (e.g. "you can expect such a problem", "there is such a problem").[15]

In the overall analysis of the collected material, a conspicuous focus placed by the teacher on the manipulation of formulas is noted. This could be an explanation of why teachers view students' poor mathematical skills as a major problem for physics courses and a limitation for learning.

Although there has not been a total lack of links between reality, theoretical models and mathematics, the type of communication encountered in the classroom is characterized by a type of physics more concerned about determining the correct mathematical formula to use. The actual ability of students to understand physical principles remains hidden.

The most positive moments, in which the type of communication has changed,

[14]It is possible that a clarification on how a formal analogy between formulas implies a different meaning would have deserved a few more words than merely defining the thing as "a bit tricky", although of course this is not trivial and the teacher may have been caught off guard.

[15]Hansson 2015, pp. 13-14.

were those in which qualitative problems were proposed, such as those that characterize the *Force Concept Inventory*.

The article concludes by declaring the need for similar studies and hoping for a progressive change in the type of communication that takes place in the classroom.

To remain in the image of the triangle proposed by the researchers, let's imagine an isosceles triangle based on reality and theoretical models and a disproportionate height with mathematics as its vertex. As a result, the link between reality and theoretical models is relatively short, and impoverished. It also follows that we are far from using mathematics as a language, we are instead faced with a mathematics devoid of meaning, the mere management of a vast collection of formulas.[16]

[16]We can mention that even in mathematics education there can be a problem of the same type, essentially when the algorithmic aspect is privileged or even the teacher penalizes the use of algorithms other than those suggested by him, even if correct.

2.4 Formulas as an epistemological tool

As we have already observed, in physics education equations are often reduced to computational tools to solve typical problems, while, almost on the contrary, in the field of research they have a role linked to the formulation of new theories.[17] As Karam suggests in his article *Introduction of the Thematic Issue on the Interplay of Physics and Mathematics*,[18] mathematics can be an epistemological tool; it allows scientific knowledge but also allows us to understand what logic is intrinsic to scientific knowledge and its development.

> When thinking about possible questions to ask to equations, one can probably distinguish between "Hows'" and '"Whys'". If, for instance, the centripetal acceleration equation ($ac = v2/r$) is presented in instruction, a teacher can (and should) focus on its meaning and investigate how the variables relate to each other (e.g., what happens to the acceleration when the velocity is doubled). But there is another important dimension that is linked to asking *why* the equation has its particular structure[19].

The ability to interrogate an equation is important to understand everything it can tell us. There is another aspect linked to this, often equally overlooked: questioning the origin of the equation we are considering. In physics lessons it is not unusual to find a naïve inductive approach, where equations are treated as mere descriptions of regular phenomena. The passage from experimental data to algebraic expression is often presented as trivial, despite the fact that they are two separate epistemological domains. Just like routines in treating equations as computational tools, routines for obtaining equations inductively (for example Worksheets) are something that deprives formulas of their educational potential[20].

In summary we could say that we would like that the way to look at equations in the didactic field was more similar to the way of considering the equations in the research field; i.e. tools that are a constitutive part of the cognitive process, rich in meaning, with the awareness that they are neither neutral nor fallen from the sky. Very well, but from a practical point of view, how can we proceed in this direction? Karam suggests two ways:

1. Talking in the classroom about the interplay between mathematics and physics from a historical and philosophical point of view. [21]

[17] Karam, 2015 b.
[18] Karam, 2015 a.
[19] Ibidem.
[20] Karam, 2015 b.
[21] We add: certainly not in the terms in which it is spoken of in a scientific journal, but it is important to break the taboo that certain topics are too difficult to be addressed. The real difficulty is more for the teacher than for the students, and lies in choosing understandable words and appropriate examples.

2. Expand the repertoire of possible justifications for the most important physical equations encountered in school.

In this eye we will consider a study published by Bagno, Berger and Eylon in 2008, *Meeting the challenge of students' understanding of formulae in high-school physics: a learning tool* [22]. Here the less virtuous approach to formulas, which we previously defined as "algorithmic", is labeled as "plugh-and-chug": the numbers are "plugged in" and the answers are "chegged out". But these problem-solving situations, in which students blindly search for equations, without making sense of physical reality, where do they come from?
The problem about plug-and-chug is that feel they have to do just that. As Karam simply illustrates, if we have been taught physics in a certain way it is natural that once we become teachers we will repeat the same modalities. We could say that the sins of the fathers fall on the children, unless there is an act of emancipation. And here is the proposal by Bagno et al. in this sense.

The study presents a test to which 35 high-school students were subjected, after studying kinematics and dynamics. Two formulas were submitted to them:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

and

$$\sum \vec{F} = m\vec{a}$$

Students were asked to:

1. Write down, using physics terms, the meaning of each of the components of the formula.
2. Specify the conditions under which the formula can be applied
3. Show that the units of measure are the same on both sides of the equation.

It was found in particular that 80 % of the students had difficulties in specifying the conditions under which the formula can be applied.

in the formula $\sum \vec{a} = m\vec{a}$, many students said that "The second law deals only with objects moving with constant acceleration".

Difficulties emerged in manipulating the units of a formula: 67 % of the students had difficulties showing that the units on the right side of a formula are identical to the units on the left side.
In particular analyzing the formula $x = x_0 + v_0 t + \frac{1}{2}at^2$, the vast majority of the students did not succeed in showing that the units on both sides of the formula are metres [m]. A few students managed to reach the relation

$$[m] = [m] + \frac{[m]}{2}$$

[22] Bagno, 2008

but could not proceed any further.

Analyzing the formula $\sum \vec{F} = m\vec{a}$, many students could not show that the units on both sides of the formula are newtons (N).

Following this test, the researchers, in collaboration with a group of teachers from the same school, developed an activity that was called "interpretation of a formula". Given an equation, students are asked to represent the relationships between its components in different ways and to identify particular houses. Only later they are asked to apply it in school-type problems and real-life scenarios.

The activity takes place in five phases:

1. An individual work, in which the students, guided by a set of tasks, explicitly elicit their knowledge about the formula.
2. The students work in small groups, on the same set of tasks, evaluate their individual work, add new ideas, and reach a consensus (or disagreement).
3. A class discussion, in which a representative of each group presents the group's consensus; all the issues raised in the group work are discussed, under the guidance of the teacher, and a classroom summary is formulated.
4. Homework on applications, in which the students use the formula in other formal learning experiences and real-life scenarios.
5. Individual reflection, in which each student individually accounts for what he/she has learned in the previous four phases and identifies what still remains unclear. It is useful to carry out a discussion in the following lesson about students' individual reflections.

The activity was implemented in 8 classes with 140 high-school students. Most of the students involved gave positive feedback. There was a general improvement in identifying the conditions under which a formula can be applied and in recognizing particular cases. For example, about the formula

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

one student reported learning that it can only be used when acceleration is constant. More interestingly, another student, regarding

$$x_1 - x_0 = 0$$

as a special case of the formula

$$v^2 = v_0^2 + 2a(x_t - x_0)$$

stated: "before this activity, I did not know that the meaning of $(x_t - x_0) = 0$ is that the objec returns to its initial position with inverse velocity."

We point out the study by Kneubil and Robilotta, *Physics Teaching: Mathematics as an Epistemological Tool*[23], which fits perfectly with the framework we have drawn. This article focuses on a didactic approach to Coulomb's law and its theoretical premises concern philosophy of education, essentially saying many of the things we have already observed but with a much deeper and more elegant argument. On the non-separability of mathematics and physics (which, however, as we will be able to observe, does not mean identity at all):

> Of course, it would be desirable to base educational action on a clear understanding of patterns realized in the relationship between Physics and Mathematics. However, the continuity between these two subjects makes the construction of comprehensive philosophical or epistemological discourses extremely difficult. The situation is similar to that of describing the tones of a sunset in a dry and dusty landscape. Although one may notice that red, yellow, and blue do coexist, and an overall pattern may be perceived, the multitude of subtle variations prevents the full description. Moreover, and what may be a more serious problem, the tones of the sunset change quickly with time, some of them fade away, whereas others show up. The same happens with Mathematics within the Physics community.

Then there is the theme of an epistemological approach to student errors, a theme that is also raised in some way in the Force Concept Inventory. Errors cannot be reduced to "deviances" simply to be corrected. The errors are inserted in an epistemological framework (not always coherent but in a certain with its own dignity), it is therefore the student's epistemological framework that must be modified. In this regard, Kneubil and Robilotta propose a beautiful quote by the philosopher of science Gaston Bachelard, which in turn we propose again and on which, for the moment, we stop.

> Pour le savant, la connaissance sort de l'ignorance comme la lumière sort des ténèbres. Le savant ne voit pas que l'ignorance est un tissu d'erreurs positives, tenaces, solidaires. Il ne se rend compte que les ténèbres spirituelles ont une structure et que, dans ces conditions, toute expérience objective correcte doit toujours determiner la correction d'une erreur subjective. Mais on ne détruite pas les erreurs une a une facilement. Elles sont coordonnés. L'esprit scientifique ne peut pas se constituer qu'en détruisant l'esprit non scientifique. Trop souvent le savant se confie à une pédagogie fractionnée alors que l'esprit scientifique devrait viser à une reforme subjective totale.[24]

[23] Kneubil, 2015.

[24] "For the scientist, knowledge comes out of ignorance as light comes out of darkness. The scientist does not see that ignorance is a tissue of positive, stubborn, solidary errors. The only realizes that spiritual darkness has a structure and that, under these conditions, any correct objective experience must always determine the correction of a subjective error. But you don't easily destroy mistakes one by one. They are coordinated. The scientific mind cannot be formed without destroying the unscientific mind. Too often the scientist confides in a fragmented pedagogy while the scientific spirit should aim at a total subjective reform". Bachelard, 1940.

2.5 Dimensional Analysis and "anchor equation"

We want to continue the discussion on the functions of equations in physics education considering some concrete tools classroom teaching. We rely on two Redish papers, *Using Math in Physics: 1. Dimensional Analysis* and *Using Math in Physics: 3. Anchor equations.*

Learning physics is difficult because the understanding of physical knowledge, often not intuitive, is compounded by the need for a casual use of mathematical symbology. We observe that this is a specific problem, different from those encountered in the first years of high school studying mathematics, where the main challenges are logical reasoning, often linked to the resolution of geometric problems, and the manipulation of a formal language; but rarely in basic mathematics (this is no longer true in the final years of high school) the two objectives to be achieved, logical reasoning and technical ability in manipulating the formulas, are proposed simultaneously, unless ad hoc exercises are proposed. In the study of physics, the coexistence of two levels, that of knowledge and that of form, is structural from the outset. Let us consider two specific resources to address this difficulty, from the perspective of what has been presented in the previous section.

Dimensional Analysis is often seen as a simple check tool to check for errors. If the two sides of the equation the units of measure do not coincide, you are doing something wrong. But Dimensional Analysis can also play a deeper role in understanding the subject. First of all, measures are not numbers. The symbols we use in physics represent quantities that can be assigned a number in different ways, depending on the measuring instrument and the choice of the unit of measure.

> When we write an equation containing measures, the statement that two things are equal means that they match physically, not that they have the same number. They will have the same numerical value only if they are expressed in the same units. As a result, dimensioned equations can look peculiar if you're only thinking about the math of pure numbers. The equation 1 in = 2.54 cm is a legitimate equation in physics because both sides represent the same physical length. The equation $x = t$ with $x = 3$ cm and $t = 3$ s is not a legitimate equation even though the numbers match. If we choose different units (as we are free to), the numbers no longer match. [25]

In a physically valid equation, the units and the numbers don't have to match, but the dimensionalities do. The concept that must reach students is that we associate a number with a symbol, but the number is not fixed, what is fixed is the property of the physical object we are describing. Dimensional analysis is also a good way to put a focus on functional dependence, direct and inverse proportionality. Having said that, the dimensional analysis remains an excellent tool to check that the equations do not contain errors.

[25] Redish, Dimensional 2021, p. 397.

By dimensional analysis we do not mean the verification of the units of measure, but of the type of unit of measure. We equate the letters in square brackets to capital letters where L stands for length, T for time, M for mass, Q for charge and Θ for temperature. For example

$$[v] = \frac{L}{T}$$

Dimensionality specifies what kind of what a symbol represents but not the specific value, dimensionality has no numbers attached to it. This makes the algebra of different from standard algebra. For example:

$$L + L = L$$

$$2L = L$$

This can create challenges, and students struggle to understand the utility of dimensional analysis if they are not asked to practice it. [26]

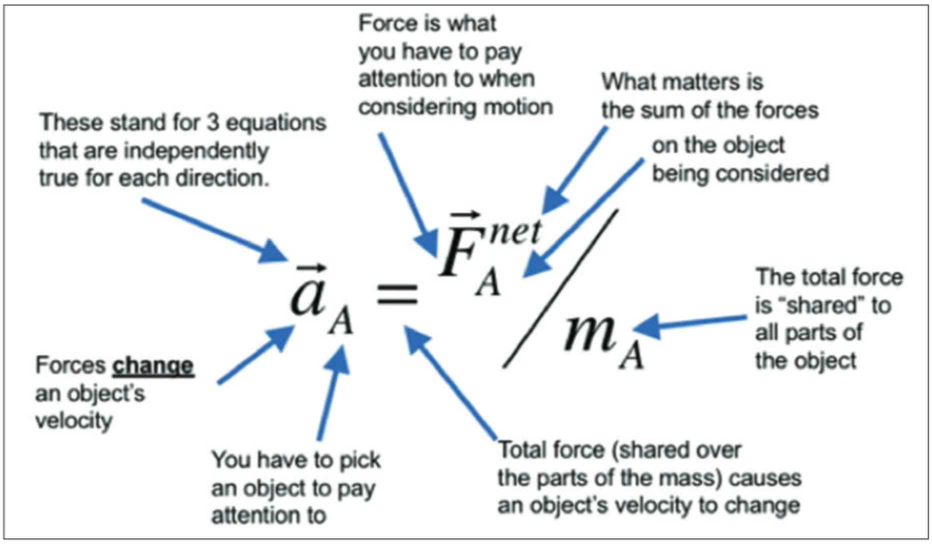
> But students have a lot of trouble with DA. It asks them to look at symbols in a way with which they have little or no experience. They're not sure that it will help them get "the answers" (which they think are numbers), so they tend to be not only unmotivated to learn it but resistive. The only way to reset their (epistemological15) expectations is to make it a part of what they are required to do and something on which they are evaluated. If we never explicitly ask them to do DA in a situation in which they are evaluated, it sends the message that we don't really think that it's important.
>
> Students' misconceptions about the kind of knowledge they are learning and what knowledge they need to bring to bear in the class are often responsible for student difficulties and resistance. I present DA in the first few classes. To show my students that I care about DA early in the class, my weekly quizzes have a DA problem in most weeks. I also give my students DA problems at various points through the class, especially when a concept with a new dimensionality is introduced. By the end of the year, a significant fraction of my students mention DA as one of the important things they have learned. [27]

Alongside dimensional analysis, Redish describes another useful teaching tool. It is in some way a possible development of what we have seen in the study by Bagno, Berger and Eylon and which is basically something that it is also possible to achieve independently in one's own teaching experience. In physics, some critical equations can help synthesize a a lot of knowledge on a content topic. Redisch refers to these fundamental equations as anchor equations to emphasize them special role

[26] Redish proposes an example of a quiz question:
The measure of the strength of an electric dipole is the dipole moment p. The magnitude of the electric force exerted by a dipole on a charge q a distance r away from it is given by (if the dipole is correctly oriented) $Fp \to q = k_C qp/r^3$. What is [p] (the dimensionality of the dipole moment)? Express your answer in terms of the dimensions M, L, T, Q, Θ.

[27] Redish, Dimensional 2021, pp. 339-400

as stable fixed points in the organization of knowledge. A typical example is of course Newton's second law, of which Redish gives this exhaustive representation:



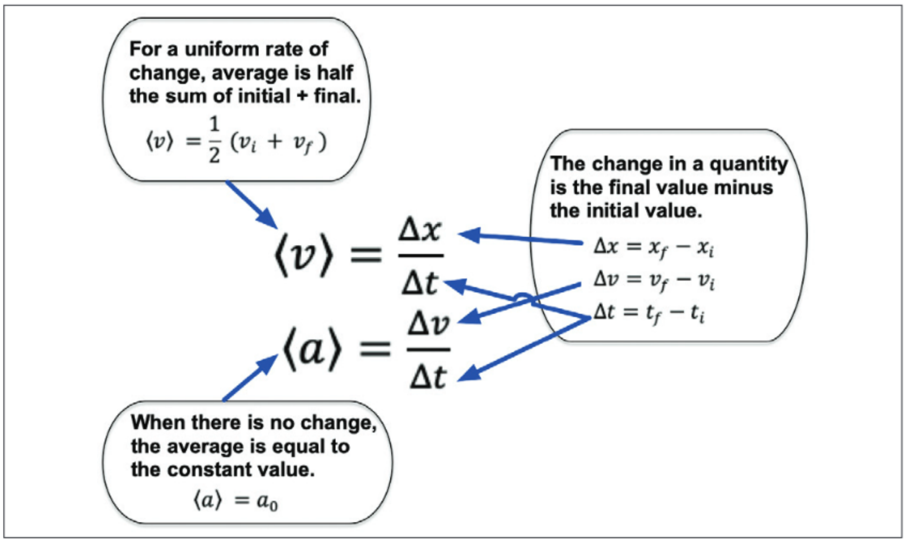
It is interesting to read Redish's narration of how he realized the importance of the way in which Newton's second law is presented, in the heart of his didactic experience:

> As trained physicists who blend physics and math and can unpack the knowledge coded in an equation, we often use a shorthand that hides the content being called upon and makes an equation look like a computational tool to be memorized. Newton's second law is a great example of this. It's the fundamental powerful principle (anchor equation) that underlies all of our understanding of classical motion at scales from the molecular to the galactic, at speeds up to a significant fraction of the speed of light. It's the principle that organizes all our knowledge of motion. And yet... Before I realized all this, I had a tendency to just write "F = ma" for Newton's second law. I sat up and took notice, howev- er, when, in one of my physics for life sciences classes, after my discussion of springs, a student asked, "Professor, what's the difference between F = kx and F = ma?" All of a sudden a lot of student behavior that never made sense to me fell into place. Why would they use F = ma for each of the different forces in a problem? Why would they tell me acceleration caused forces? And how would they ever forget it? Well, of course one reason is they were not making the blend, just using each equation as a calculational tool. But a second reason was that they were failing to identify one of the two equations as a core principle (and the other as a crude toy model).[28]

Other examples are the work-energy theorem, Maxwell's equations, the Schrödinger equation, the already mentioned Coulomb's law, Einstein's equations of general relativity.

[28] Redisch, Anchor 2021, p. 601.

Emphasizing the importance of fundamental equations is one way of countering the deleterious tendency of textbooks to put a collection of equations at the end of the chapter, giving the misleading message that these many equations are independent calculational tools. Emphasizing the importance of fundamental equations is one way of countering the deleterious tendency of textbooks to put a collection of equations at the end of the chapter, giving the misleading message that these many equations are independent calculational tools. A typical example is with the equations of kinematics, where all particular cases can be obtained from the law of uniformly accelerated motion, or, as Redish suggests, from the same definitions of speed and acceleration.



To get students to value anchor equations, it's important to offer problems easy to set up using an anchor equation and a few straightforward manipulations, rather than ones that are easy to simply put numbers into a memorized equation and calculate. If this type of ideas reaches students, they will start taking notes not mechanically but organizing their notes around them by organizing them around some key concepts, such as anchor equations. Their exam performance will improve along with the perception that "there really isn't that much to learn", even though there is, but it seems less if it can be developed from a small set of "things to know".

Chapter 3

A teaching module within the IDENTITIES project

3.1 Introduction to the module and the IDENTITIES project

After a historical overview of the concept of demonstration and an overview of didactics of physics, in this last and third chapter we will consider something more concretely close to the world of teaching.

In the present chapter the module on parabola and parabolic motion is presented. It consisted in eight lessons and it was implemented within a physics education course at the University of Bologna in October 2020. The name of the course is "Teaching of physics", held by Professor Olivia Levrini. The course is mainly targeted both to physics and to mathematics students who have chosen the teaching curriculum and who are likely to become teachers.

The lectures involved different experts and researchers from Professor Levrini, expert on physics education, to Professor Laura Branchetti, expert on mathematics education, to Professor Paola Fantini, a secondary school teacher and researcher. In the following sections the contents of the lessons are presented.

The topics covered, as we will see, are not far from those already addressed in this work, namely the role of proof and the interdisciplinarity between mathematics and physics, but the perspective is, or at least would like to be, more teaching-oriented. The conditional is bitterly obligatory, because as anyone who has had teaching experience in high school probably knows, the most splendid theoretical preparation does not prepare, and cannot prepare, for the impact with human and institutional dynamics that are often very far from the topics studied at University.

Just to give some trivial examples, a high school teacher has to confront with the difficult relationship with adolescents, for which no course in psychology or pedagogy can prepare, or the pressure exerted by the parents. Every utopian attitude inspired by the movie with Robin Williams Dead Poetry Society must be descended into a highly bureaucratic institutional context, which in all likelihood will shape the teacher himself. Bringing innovations can lead to rejections, both on the part of the "system" and on the part of students themselves. If they have become accustomed to specific learning methods, for example a mnemonic type, they could not address favorably on novelties, since they, stimulating a different kind of reasoning, could lead to an increaseing fatigue.

With this in mind, two aspects make this chapter at least in part closer to the reality of the school than the previous two. The first aspect is that the content of the module are curricular topics, so school teachers are used to introduce them separately in mathematics, the parabola, and in physics, the parabolic motion. The second, perhaps even more interesting, is that we will not limit ourselves to retracing the topics covered during the module but we will also present the preliminary results of an analysis of the essays, delivered from the students before the final exam, with the aim of investigating the main difficulties that students have when they talk about the proof (Chapter 4) Students who have the peculiarity of already thinking of themselves as future teachers but who in most cases are not chronologically distant from their school experience. Taking their feedback into consideration gives an idea, however indicative, of how our themes are perceived by those who will be the teachers of tomorrow and who were at school a few years ago.

The module and this thesis itself are part of the IDENTITIES project (https://identitiesproject.e an Erasmus + project coor- dinated by the University of Bologna. The project also involves the University of Parma, Crete, Montpellier and Barcelona. The name stands for: Integrate Disciplines to Elaborate Novel Teaching approaches to InTer-disciplinarity and In- novate pre-service teacher Edu- cation for STEM challenges. STEM is another acronym (people involved in science education love acronyms) which stands for: Science, Technology, Engineering and Mathematics. The main objective of IDEN- TITIES project is the design of novel teaching approaches on interdisciplinarity in science and mathematics to innovate pre-service teacher education for contem- porary challenges. The interdisciplinary topics, focus of the research, are both emergent advanced STEM themes (i.e. climate change, artificial intelligence, nano- technologies) and curricular interdisciplinary topics (i.e. cryptography, parabola, non-Euclidean geometry, and gravitation). These themes will be the contexts to explore inter-multi-trans-disciplinary forms of knowledge organization and to de- sign classroom activities and new models of co-teaching.

3.2 Interdisciplinarity and epistemological activators

The acronym IDENTITIES naturally refers to the concept of identity, in particular to the identity of the various disciplines. More specifically, it is meant that the interaction between different disciplines is something that strengthens, rather than attenuates, specific identities. Interdisciplinarity does not mean breaking down the boundaries of the different disciplines, on the contrary we assume that disciplines are necessary and that each has a specific role. The origin of the term discipline comes from the Latin "discere", that is referred to learning. So, the disciplines are a body of knowledge and skills that ground their roots into the educational necessity to re-organise knowledge to teaching and learning it. [1] The canonical division of physics into mechanics, thermodynamics, electromagnetism, etc., is an organization that was born at the end of the nineteenth century to effectively transmit knowledge from one generation to another. The issue of interdisciplinarity has been addressed in recent years in the teaching theory of mathematics and physics to prevent the risk of trivializations; the risk is to use mathematics as a mere calculation and formalization tool in physics and physics as a mere application field of mathematics (REF). The relationship between the two disciplines, both in the scientific revolution of the seventeenth century, but also the revolution of the late nineteenth century that led to the rewriting of the foundations of the two disciplines, sees them deeply connected. Is it possible to show in class that mathematics is something that introduces new forms of reasoning into physics? How can we highlight the structural rather than an instrumental role of mathematics in physics? Looking for examples, the role of geometry (of geometries) in the special theory of relativity is certainly visible. Using one mathematics instead of another brings into the physics of specific epistemological assumptions.

In order to value the role of mathematics in physics and vice versa, two important historical case were chosen. On one hand the discovery of the parabolic shape of the trajectory of a projectile that led to the establishment of Physics as a discipline and, on the other, the unification of the conics carried out by Apollonio that led to the birth of projective geometry. Both the episodes are exemplar for reflecting on the nature of interdisciplinarity and on the structural role that both the disciplines played.

In order to reflect on it, the research group elaborated a construct, that is called "epistemological activators". For epistemological activator they mean "kinds of themes / activities / ideas that can foster the activation of epistemological reflections on the nature of knowledge and science itself by either organizing knowledge on a higher abstractive level or setting a context where specific ideas can become key concepts" (Ravaioli, 2020).

In the module, symmetry and proof were reconceptualized as "epistemological activators", in order to guide students to a) reflect on the disciplines' identities and b) the interdisciplinary "contamination" that characterize the disciplines' evolution. The presence of symmetry in the representation of motion led to an idea of motion as we conceive today, overcoming the Aristotelian conception of motion

[1] Alvarogonzález, 2011, pp. 387-403.

as a a succession of violent and natural motions. This episode is representative also because it marked the shift "from the world of the about to the universe of precision" (Koyre, 1967), in which the mathematics from the celestial world descends into the terrestrial world expanding the interpretative schemes that, until then, were used to represent the earthly motions: straight lines and arcs of circumference. In fact, the works of first Guidobaldo dal Monte with the introduction of the experiment as practice and method of physics, and then of Galileo Galilei, who proved the parabolic shape of the projectile's trajectory, led to consider the parabola as a possible interpretative scheme.

Symmetry and proof were reconceptualized as "boundary object" (Akkerman & Bakker, 2011). For boundary objects Akkerman and Bakker mean "artifacts that articulate meaning and address multiple perspectives.

...

The boundary object belongs to both one world and another. It is precisely this feature that seems to explain how the boundary divides as well as connects sides. However, the boundary also reflects a nobody's land, belonging to neither one nor the other world.

...

On one hand, they enact the boundary by addressing and articulating meanings and perspectives of various intersecting worlds. At the same time, these objects move beyond the boundary in that they have an unspecified quality of their own (neither–nor)" (Akkerman & Bakker, 2011). So boundary objects are characterize by an intrinsic ambiguity that allow us to enact a multivoicedness (both–and) and the unspecified quality (neither–nor) of boundaries, creating a need for dialogue between disciplines and people, "in which meanings have to be negotiated and from which something new may emerge." (Ibidem, 2011).

In this perspective, symmetry and proof were reconceptualized as boundary object, since their characteristic of both-and and neither-nor in these historical episodes, and their educational potential was exploited to explore both the identity aspects of mathematics and physics and the ways in which the two disciplines dialogues, so as a way to overcome the boundaries and to recognize the interdisciplinary relationship and co-evolution of these two disciplines.

In the following, the content of the module are introduced in a deeper way.

3.3 Basic connotations of a scientific text

Linguistics provides tools for assessing the consistency of a text. These tools help us to analyse a text on, at least, three different level:[2]

1. Textual level, that allow to observe the more general structure of the text and the thematic progression, and how it describes the interweaving of mathematics and physics. The thematic progression operatively concerns the use of connectives, repetition, implicit content, etc. This can highlight how different parts of the text communicate with each other to deliver the message as intended by the sender.

2. Syntactic level. It highlights the prevalence of coordinates or subordinates and, hence, points out the logical links between different portions of sentences and text.

3. Lexical level. It consists in the analysis of the particular terminology used, of the semantic field of words and expression in the text. It is fundamental for example to observe the scientific lexicon and its belonging to different disciplines (e.g. mathematics and physics).

On the textual level logically connected parts of text can be linked by connectives, or the links can be implicit and left to the reader. The implicit content can be of two types. One type is what is taken for granted, the other is what is implicitly suggested, for example through juxtapositions. The implicit contents are often potentially carrier of misunderstandings, let us see an example of bad juxtaposition:

> "Einstein wanted to reconcile the theory of electromagnetism with the principle of relativity already formulated by Galileo: all inertial reference systems in motion with respect to each other are equivalent and none can be considered in motion or at rest absolutely".

Reading this text it is not clear (unless you already know it) if the sentence after the colon refers to the reconciliation between electromagnetism and Galieo's principle or if it is only Galileo's statement.

So, what are the specific characteristics of a scientific text? We can note a prevalence of informative and expository aspects to the detriment of descriptive and narrative aspects, which are nevertheless present. It is a text that has the function of informing the recipient and enriching his knowledge about a specific topic. The content must be well organized. The text must make explicit what the recipient is not required to know before reading the text.
The language is sectorial, has its own vocabulary, is aimed at a small community, which can be made up of students or scientists. Each word corresponds to a single object, such as speed, force, boson. The scientific text prefers impersonal forms, nouns prevail over verbs, there is an unusual density of meaning compared to other types of texts. A textbook has a greater rigidity than a popular text or a scientific

[2]Polverini, 2022.

article, since in an educational context the informative function often prevails over the argumentative one. A scientific text must convince the scientific community, a textbook exposes something consolidated.

The tools introduced can be used also to investigate from a linguistic point of view the historical texts and textbooks. In this case, the tools were used to point out some differences and similarities between Walker's textbook [3] and the famous text *Discorsi e dimostrazioni matematiche intorno à due nuove scienze attenenti alla mecanica & i movimenti locali* written by Galileo Galilei. [4]

We can observe how in Galileo's work a real dialogue takes place only in the first two days. On the third and fourth day Salviati reads a treatise in Latin on motion. The use of the vernacular is in itself an innovation by Galileo and the form of the dialogue, of Platonic tradition, favours the identification of the reader. The text is not a textbook but a hybrid between a scientific text and a work of dissemination. It is of course a work aimed at a cultured public. Walker, for his part, in a scholastic perspective, pays particular attention to scientific language and to take nothing for implied. He specifies that a projectile does not mean (only) the ammunition of a firearm, but any object that can be thrown (i.e. projected) and which is therefore subject only to the force of gravity. In both Walker and Galileo the structure of paragraphs and chapters is described, to allow the reader to orient himself.

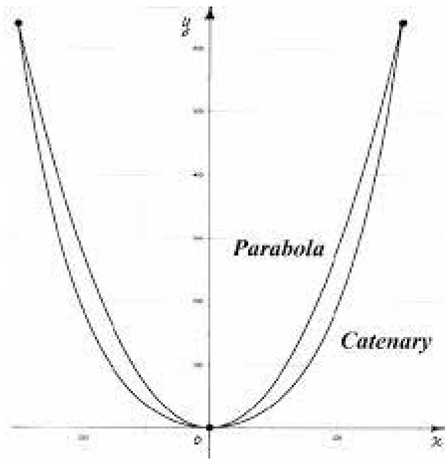
[3] Walker, 2017, pp. 88−116.

[4] Galileo, Discurses.

3.4 Introduction to Galileo's proof of parabolic motion

The study of the motion of projectiles, has historically marked the transition from qualitative to quantitative arguments in physics. In the Aristotelian, pre-Galilean perspective, mathematics, in particular geometry, was considered suitable for describing the celestial world while it was not suitable to illustrate the sublunar world, full of imperfections. The idea of composing two different motions, the uniform rectilinear one and the uniformly accelerated one, thinking them independently and not in opposition, was revolutionary. The only motions conceived by Aristotle were the rectilinear one and the circular one. And so even in the sixteenth century parabola was not part of the interpretative schemes. Modeling the motion of a projectile as a parabolic motion is in itself an important epistemological leap. Change occurs when science has to respond to practical and concrete needs such as the study of ballistics; science as we conceive it implies the passage from a contemplative dimension to an applicative and active dimension.

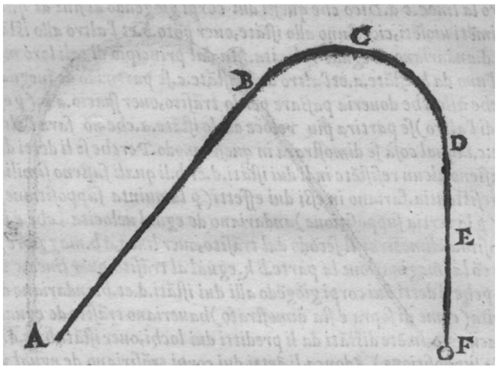
The experimental method is an intertwining between theory and technique, in which the observation of nature is not enough. In an experiment a phenomenon is isolated, it is "purified", made to resemble an ideal situation which is not the phenomenon itself but what makes it understandable, studyable, and measurable. Guidobaldo Del Monte had made empirical observations on the trajectory of a projectile by modeling it with a catenary.



In his notes he describes an experiment, in which a ball dipped in ink is thrown on an inclined plane, drawing its own trajectory. This allows to observe the symmetry in the ascent and descent phase. The experiment is conceived as a result of interpretative conjectures, and is in support of the same. A similar method is described by Galileo as a "wonderful" way to draw exactly a parabola. The creation of a mathematical object is somehow placed on the same level as the description of a physical phenomenon. For Galileo the geometric form is concretely realized in matter, the parabola is the trajectory. The new physics is a reification of celestial mechanics in nature, it is a geometric physics and a physical geometry.

3.5 Two proofs in two different contexts: a modern textbook and Galileo's Discourses

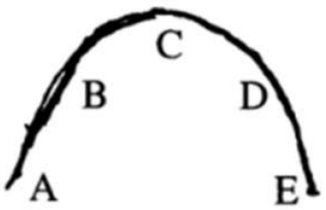
As we anticipated before, the proof, in the educational reconstruction carried out, can be thought as an "epistemological activator". To understand how much the Aristotelian vision permeated the vision of the world up to the sixteenth century, it is interesting to consider the representation given by Tartaglia.



The perception of phenomena is conditioned by one's own cultural substrate. As stressed by Renn

> The first part of the trajectory was conceived by Tartaglia as reflecting the initially dominant role of the violent motion, whereas the last straight part is in accord with the eventual dominance of the projectile's weight over the violent motion and the tendency to reach the center of the earth. The curved middle part might have been conceived of as a mixed motion compounded of both violent motion in the original direction and natural motion vertical downward.[5]

On the other hand, In the notebook of Guidobaldo Del Monte, as we have seen, the representation of the motion shows a symmetrical shape, which implies that natural and violent motions can be composed and not only followed, and that for the trajectory of motion there are possible shapes other than rectilinear or curvilinear ones.



However, the conception of the opposition between natural and violent movements persists. Guidobaldo observed that the motion of a projectile is of the curvilinear type, and he drew a curve that could represent it, but he did not prove

[5] Renn, 2001

that the trajectory of the motion corresponds to a particular curve. Before getting to the heart of Galileo's proof, let us consider following proof of parabolic motion, taken from a modern common high school textbook by James Walker[6].

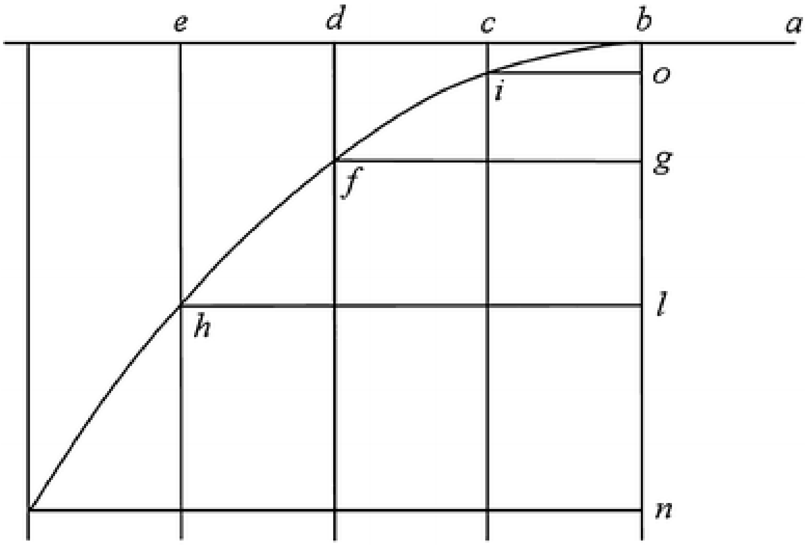
> What is the shape of the curvilinear trajectory of the projectile thrown horizontally? We can find it by combining the equations $x = v_0 t$ and $y = h - \frac{1}{2} - gt^2$ so as to express y as a function of x.
>
> First of all from the equation $x = v_0 t$ we derive the time.
>
> $$t = \frac{x}{v_0}$$
>
> Then we substitute this result in the equation $y = h - \frac{1}{2}gt^2$ and so we eliminate t.
>
> $$y = h - \frac{1}{2}g\left(\frac{x}{v_0}\right)^2 = h - \frac{g}{2v_0^2}x^2$$
>
> We observe that y is an equation of the type $y = a + bx^2$, where $a = h =$ constant and $b = \dfrac{-g}{2v_0^2} =$ constant. This is the equation of a parabola with concavity towards the bottom and represents the characteristic shape of the trajectory of the motion of a projectile.

Faced with the simplicity of these few lines, what is the meaning of all the reconstruction we are doing? Perhaps, among other things, also to restore the historical importance of the birth of science. Bertolt Brecht, in his Life of Galileo, when it is discovered that the Moon is made of rocks and mountains, and not of celestial material, puts these words in the mouth of the scientist:

> Don't take your eye off the telescope, Sagredo. What you are seeing is that there is no difference between heaven and earth. Today, January 10, 1610, humanity writes in its diary: "Heaven abolished".

Let us now see Galileo's proof from *Discourses*. Galileo has yet introduced uniform motion in an axiomatic way and has already shown that in the fall of a body the space covered is proportional to the square of time.

[6] Walker, 2017



Let it be a line *ab*, placed at the top, and a body moving with uniform motion from *a* to *b*. Lacking the support of the plane at point *b*, the body performs a natural motion along the perpendicular *bn*. Let the line *be*, continuation of the line *ab*, be a measure of time, and on this line an equal number of equal portions *bd*, *cd*, *de* should be marked. From points *c*, *d*, *e*, draw equidistant lines perpendicular to *bn*. On the first of these, take an arbitrary part *ci*. On the next line take one four times greater, on the third one nine times greater, *df*, and so on on the other lines according to the proportion of the squares of the portions *cd*, *db*, *de*, in duplicate proportions of the same.
Furthermore, if the body, which from *b* towards *c* with uniform motion, is added a downward motion of quantity *ci*, in time *bc* the body will be in the extreme *i*.
Furthermore, if the body, which from *b* towards *c* with uniform motion, is added a downward motion of quantity *ci*, in time *bc* the body will be in the extreme *i*. Continuing to move, in a time *bd*, that is in a time double of *bc*, it will have fallen by a space four times greater than the space *ci*; in fact we have shown in the first treatise that the spaces traveled by a heavy person, with naturally accelerated motion, are in double proportion of the times. Similarly, the next space *eh*, covered in time *be*, will be nine times greater; it will therefore be evident that the spaces *eh*, *df*, *ci*, stand together as the squares of the lines *eb*, *db*, *cb*. Now lead from the points *io*, f, h the lines *io*, fg, *hl*, equidistant from the same *eb*: the lines *hl*, fg, i will be equal, one by one, to the lines *eb*, *db*, *cb*; and likewise the lines *bo*, *bg*, *bl* will be equal to the lines *ci* , *df*, *eh*; moreover the square of *hl* will be proportional to the square of fg as the line *lb* is proportional to *bg*, and the square of fg will be proportional to the square of *io* as *gb* is proportional to *bo*; therefore, the points i, f, h are on one and the same parabolic line. Similarly, it will be shown that, taking any number of equal parts of time of any magnitude, the points which the moving body of such a compound motion will occupy in those times will be found on the same parabolic

line. It is therefore clear what we set out to demonstrate.[7]

[7] Galilei, Discourses

3.6 Argumentation and proof

The difference between Walker's and Galileo's proofs leads us to ask ourselves: what can be the usefulness of having several proofs of the same proposition? What works and what may not work on an educational level. Let's take a step back and consider the ministerial indications for the teaching of mathematics in high schools; here a particular formative aspect of the proof is highlighted: the student must be able to argue his own convictions with examples and concatenations of observations, he must also be able to change his opinion by recognizing the validity of a logical argument.
Passing from the generic concept of argumentation to the more technical one of mathematical proof, we are asked to focus on the modalities of the axiomatic-deductive system.
In teaching theory, an argument is a path that starts with some data and arrives at a conclusion, using some warrant which in turn is based on a foundation. In the context of argumentation, the foundation can also be a personal experience or a sensitive experience, in the context of demonstration the foundation must be a theory.
In introducing students to the concept of proof, there may be difficulties in making them perceive its necessity or validity. In the case of demonstrations of intuitive truths, the proof can be perceived as superfluous if what one wants to prove seems evident already in looking at the figure. Conversely, when demonstrations of less intuitive facts are presented, the demonstration can be unconvincing or perhaps too formal.

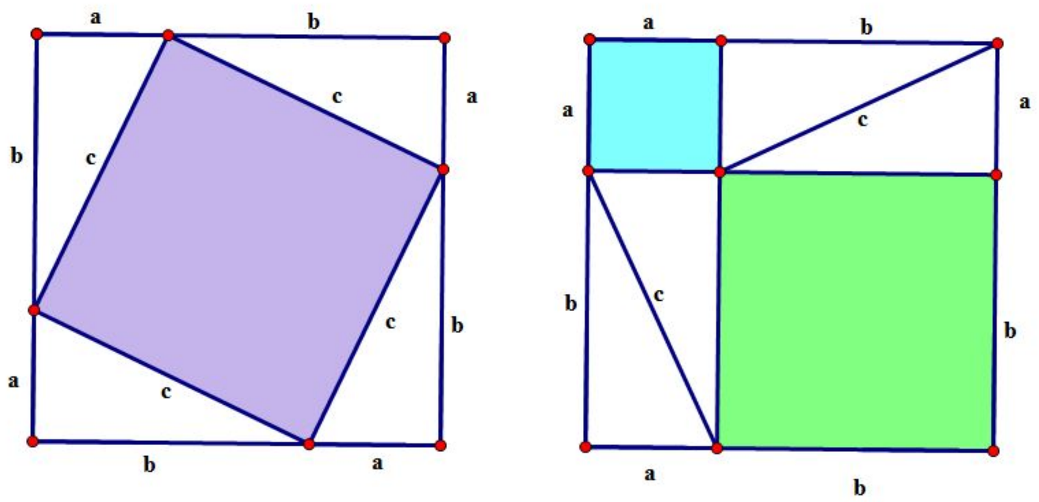
To overcome this paradox it is useful to reflect on the different functions of a proof.[8]

Let us take as a reference probably the most famous theorem of all mathematics: Pythagoras' theorem, of which there are a multitude of proofs after the one provided by Euclid in the first book of the Elements. Why give further proofs, if the theorem had already been proved? Simplifying, we can observe that there are several macro typologies of proofs that perform different functions.
We have "visual" proofs, of which an example is that of the arrangement, which is based on the idea of equidecomposable figures, and could be accepted by Euclid[9]. The function of this type of proof is to convince. It is easier to "see" that the Pythagorean theorem is true by observing such a proof than that of the Elements, where previous results on parallelograms are used [10].

[8] Mariotti, 2006, pp. 173-204.

[9] The Chinese proof we saw in chapter one is also of this type.

[10] Or using what in the scholastic tradition is called Euclid's Theorem or Euclid's First Theorem, which in the Elements is nothing else than the first part of proposition I.47, known as Pythagoras Theorem.



Another type of proof is that which uses results subsequent to the Pythagoras theorem, such as the formula for the distance between two points or trigonometric relations. Since these are based on the validity of the Pythagoras' theorem, their function is not to create new knowledge when to relate different parts of the mathematical building, convincing us of already known results through other results. It is also a reassurance that the building is consistent.
Didactic theory shows that, if formally there is no difference between accepting the validity of a proof and accepting the validity of an affirmation guaranteed by a proof. But in the students' perception there is a difference, and the need emerges for an intuitive acceptance of the validity of a statement. Therefore if on the one hand it is necessary to carry on the exposition of the axiomatic-deductive method, with its essential formal and rigorous components, on the other hand this must be accompanied, at least orally, by a more immediate and understandable discourse.

It is possible to characterize a mathematical theorem as consisting of a statement, a proof, and the fact that the relationship between statement and proof makes sense in a specific theoretical context. The same proof can be valid in one theory and not be valid in another. Added to this are the inference rules, which can be explicit or implicit.

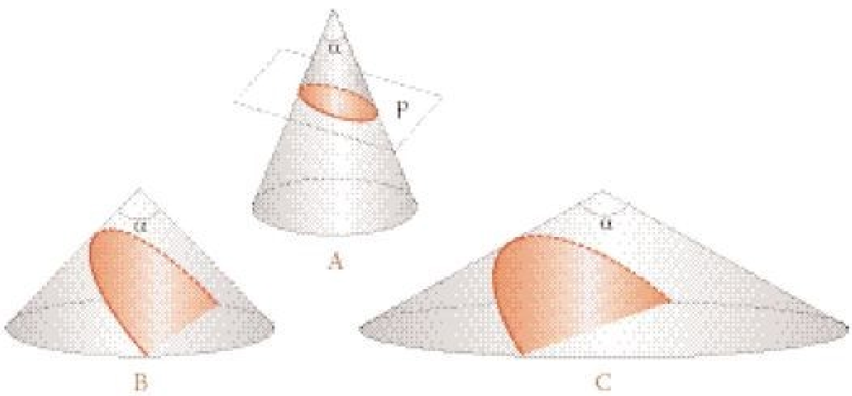
We can say that a proof has an "argumentative" role when its goal is to convince a result through a concatenation of arguments. A function has a "relational" role when its goal is to show what other results it depends on, thus relating various elements of a theory. These are the two roles that are exercised in teaching. In research, the main function of demonstrations is to generate new knowledge. These categories could also be used to characterize Walker and Galileo proofs.

3.7 A historical glimpse on conics

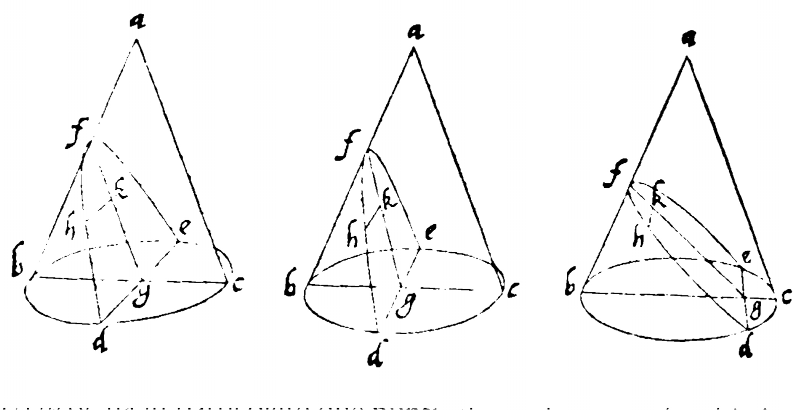
It is important to note that the characterization of the parabola used by Galileo is that given by Apollonius. It is a characterization that in some way unites the two main definitions of conics: that of a conic section, therefore determined by the intersection of a cone and a plane, which therefore depends on the properties of these two (fixing one and varying the other we get all the conics) and that of an object that lives in a plane, which therefore depends on a single equation, or property.

Although in Apollonius there is of course the concept of the Cartesian plane and the formalism used by Descartes, his work can be thought of as a forerunner of analytic geometry.

In Euclid's Elements the conical sections are obtained by keeping the plane fixed and varying the shape of the cone with which it intersects.



In Apollonius the cone is fixed and the position of the plane varies.



In Apollonius the cone is generated by the rotation of a segment that has a fixed point at one end, which will be the vertex and at the other end a point that rotates on a circle. What we would call today the conic axis is called the "ordinate direction", which is what we would now call the abscissa axis. Using the theory of proportions, Apollonius identifies a fundamental relationship that characterizes the parabola, which today we would express as the proportionality between the ordinate and the square of the abscissa.

Apollonius, however, does not characterize the parabola using the concept of focus, because unlike the hyperbola and the ellipse it has one instead of two, being the second focus identifiable only in projective geometry as an improper point. The attempt to characterize all conics through their foci is carried out by Kepler, a problem that emerged in the study of the physical phenomena of reflection and refraction of light.

Back to the characterization of the motion of the projectiles, we note a methodological difference in Galileo's approach compared to that of Guidobaldo Del Monte. Guidobaldo starts from a graphic idea and tries to understand which curve you resemble. It could be a hyperbola, a catenary, but it's not essential. Galieo, on the other hand, geometrically constructs the parabola, and the construction of him has a Euclidean meaning, an existential meaning: what can be built exists.
Galileo takes up Apollonius' idea of proportioning between segments (we would say) of the abscissas and (we would say) squares of the ordinates, without having in mind the modern concept of function. The reasoning is purely geometric, it is verified that the constructed object is a parabola because it respects the definition of Apollonius. His work does not suffer from the lack of analytical geometry to be considered rigorous.

In conclusion, the rigor of Euclid, Apollonius, Descartes, Galileo, must be interpreted within their own system of knowledge, within a theory. Galileo was rigorous in mathematically proving precise physical assumptions. We also observe that the parabola in mathematics textbooks is made to coincide with his equation, but it is much more and can be characterized in many different ways and has had a decisive importance in the history of thought.

Chapter 4

A study on awareness on proof

4.1 Context and method of the study

Aim of the study Our aim was to investigate preservice teachers' attitudes and knowledge on proof in mathematics and physics education. In particular, we have tried to detect:

1. Awareness of the characteristics that define a demonstration, in general and more specifically in a didactic context.

2. Awareness of the role of mathematical proof in physics and in a physics didactic context.

Context of the study The IDENTITIED module on parabola and parabolic motion, was held within the course on "Physics teaching, theoretical aspects and experimental aspects.", at the University of Bologna. The responsible teacher was Olivia Levrini, four lectures were hold by: Paola Fantini (one lecture), high school physics teacher, Laura Branchetti (two lectures), researcher in mathematics education, and Veronica Bagaglini (one lecture), linguist.

The implementation lasted 20 hours, from the month of October to the month of November 2020. About 60 university students in physics and mathematics were involved.

At the end of the course the students were asked to deliver an essay following the some questions both on the content of the module and on the educational value of an historical and interdisciplinary approach to the content. Our analysis focuses on the answers given by 25 students (14 mathematics student and 13 phisycs students) to the following two questions:

1. What functions can a proof have and what characterizes different proofs, for example the different proofs of the Pythagorean theorem?

2. What characterizes the proof in Galileo and the proof in Walker? What role or function can these proofs have?

Aims of data analysis Are mathematics and physics university students acquainted with the construct of proof? Did the students understand the contents of the module on proof? Did they appropriate the criteria to analyze a proof in physic or mathematics education?

Methods of data analysis In the evaluation of the answers we used a rubric composed by five parameters, to each of which we assigned a score from 1 to 5.

1. Correctness. With this parameter we assessed whether the student described the different functions of the proofs as they were presented in class, associating each with the correct meaning.

2. Completeness. This parameter evaluates whether the student has cited all the functions mentioned, whether he refers to the different examples of proofs that have been proposed with respect to the Pythagoras' theorem.

3. Richness: Is the argument rich? Was the student able to present a well articulated and dense speech or is the argument excessively concise and / or simplified?

4. Re-elaboration: Does the student's answer show traces of ideas, concepts, words that obviously belong to himself? Does the articulation of the answer show signs of personal re-elaboration?

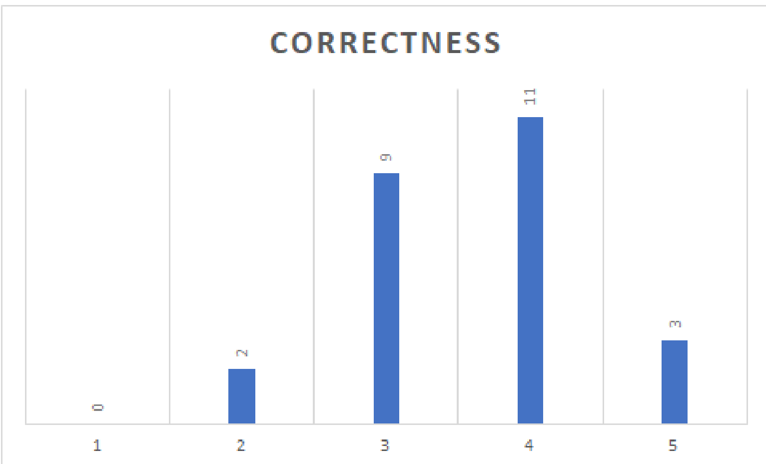
5. Consistency: In answering the second question, does the student use the criteria introduced in the first answer? Does the student mentions the different functions that a proof can have to dispel the differences between Galileo's proof and Walker's one? If he has introduced new criteria in the first answer, does he then use them when talking about the differences?

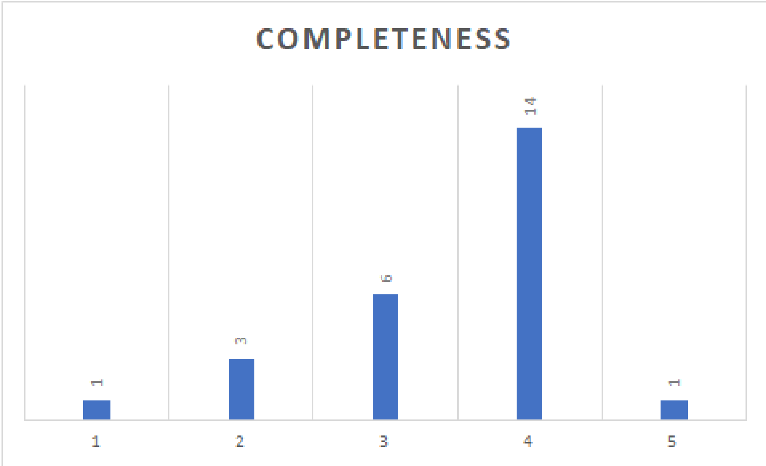
A careful evaluation of the esseays was carried out by me and Dr. Sara Satanassi, separately. Then we triangulated the outcomes of the assessment, confronting ourselves we reached a common agreement. We used different markers in the text to highlight different aspects: orange to indicate expressions that were not precise enough, red for sentences that contained conceptual errors, bold for sentences that indicated a personal re-ealboration and blue for sentences that indicate a particular consistency between answer 1 and answer 2.

Answer 1	Correcteness	Completene	Richness	Re-elaboration		
it is used to explore or reorganize knowledge, to relate assumptions and working hypotheses with the results of a theorem in order to frame the field of validity of this 'last (and later also modify it after having given the right weight to the hypotheses), to argue and explain a concept in a rigorous and convincing way. Different demonstrations of the same topic involve different objects and different paths, thus allowing everyone to adopt the style they prefer, both in making the demonstration and in listening and understanding it; in addition to the question of personal choice, the use of different demonstrations allows us to have a clear understanding of the validity field of the theories exposed and also	3	2	2	3		
Answer 2					**Coerence**	**Differences reported**
because they both start from the same hypotheses (independence of the motions, space proportional to time along x and to the square of time along y), obtain the values of x and y and realize that the relationship between y and x is quadratic in x; they differ only in that Galileo connects the points x, y graphically, explicitly obtaining a parabolic curve, while Walker connects the points x, y mathematically obtaining the equation of a parabola. On a deeper level, however, a great difference is evident, which allows us to reason on the distinction between physical proof and mathematical proof (therefore a profound difference in terms of the field of validity and the way in which new knowledge is generated). In general, it is good to remember that in physics (as in the other natural sciences) a theory is confirmed, that is, it is in agreement with the results of an experiment and observations. Mathematics, on the other hand, is a logical system based on fundamental axioms, therefore the knowledge produced is part of an internally coherent system in which the proved theorems become part of the system continuously. In light of this distinction, the difference					2	In Galieleo points are connected "graphically", in Walker "mathematically". "Physical proof" VS "Mathematical proof".

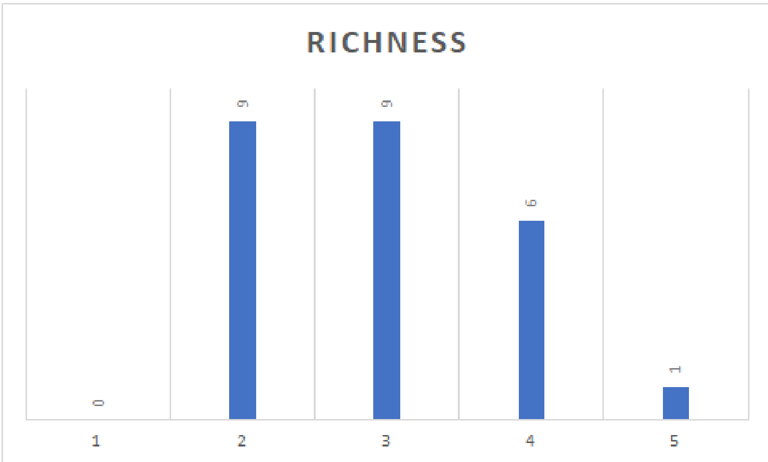
This was what the grid looked like.

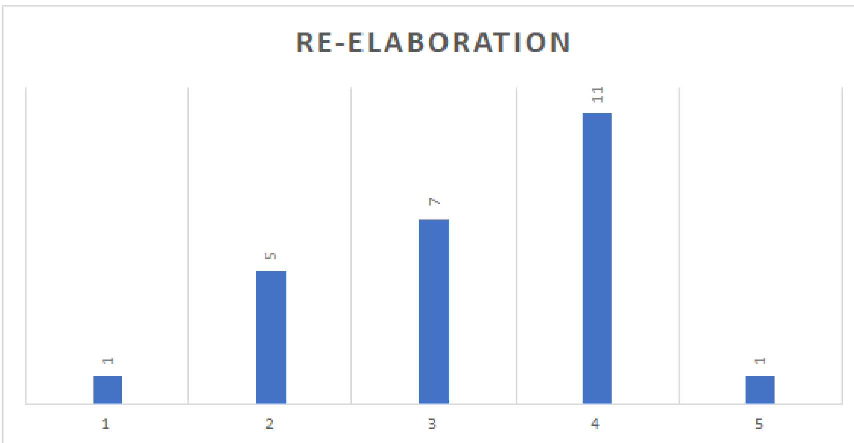
Distributions of assessments Let us now observe the distributions of the evaluations in form of histograms.

1. 

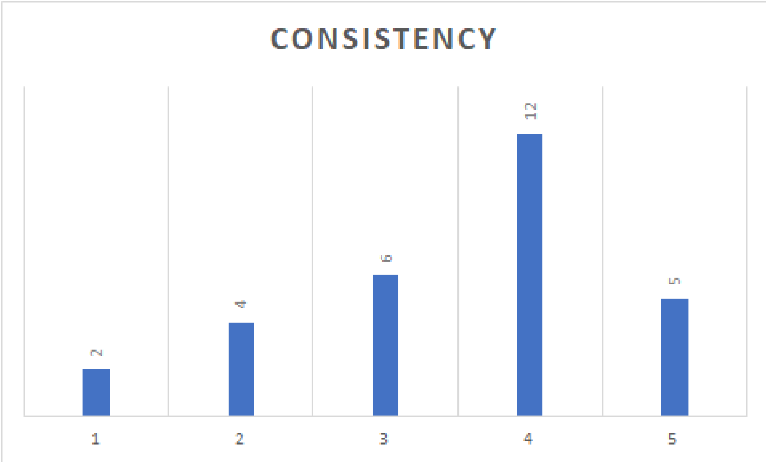
2. 

We observe that the students had a good performance in terms of correctness and completness, that is, in reporting what was explained in class.

3. 

4. 

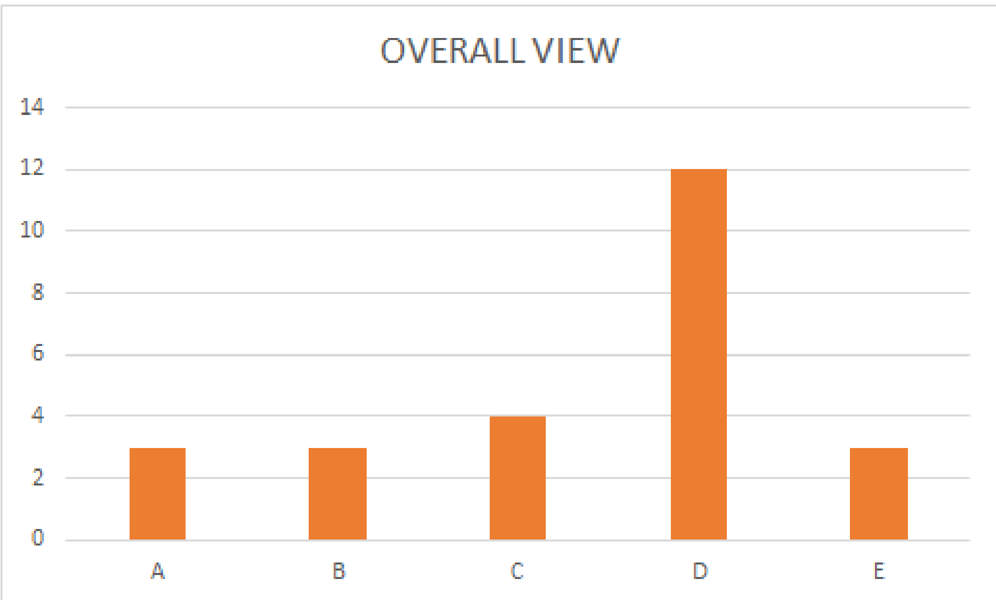
We observe a lower performance in terms of richness and re-elaboration.

5. 

A good majority of the students were consistent.

An overall picture In order to have an overall picture organize the essays in five categories:

- A. Essays very weak, ranking from 1 to 3 in all the first four criteria;
- B. Essays ranking from 4 to 5 in the first criterion on correctness, but rather week (ranking from 1 to 3) along all the other three criteria (completeness, richness,and personal re-elaboration);
- C. Essays ranking from 3 to 5 in the first two criteria, correctness and completness, and from 1 to 2 in richness and personal elaboration;
- D. Essays ranking 3 or 4 in all the criteria;
- E. Essays ranking 4 or 5 in the personal re-elaboration but rather weak (from 1 to 2) in one or more of the other criteria.



4.2 Interpretation of the data

Responses analysis Observing the histograms, although the sample size cannot give statistically significant indications, students seems to have understood the different functions of the proof and, sometimes, to critically use them to reflect on Galileo's and Walker's proof. In particular we observed a better performance in terms of correctness and completeness rather than in richness and re-elaboration. This may suggest a lack of familiarity with the concepts presented, as evidenced by another test, which we will present later.

Several students, speaking of the differences between Galieleo's and Walker's proof, have described Galileo's as based on "senses" or "perception", but also as a "physical" proof, as opposed to Walker's, "formal","algebraic" and "mathematical". Several students had the perception that Galileo's proof was not mathematically rigorous.

For example, a mathematics student wrote: *"Galileo's proof is more physical, Walker's one is purely mathematical. It could be said that Walker's proof plays a more abstract role. While Galileo's is more operational, that is, it does not serve so much to better describe what is seen, rather it provides a "recipe" useful for carrying out precise modifications"*.

Another mathematics student: *"If Galileo only mentions mathematics (speaking of a parabola), Walker only uses mathematics in his proof; this makes understanding on a physical level more difficult. It can be said that Galileo's proof is more effective on a cognitive level because it leads to reasoning and asking questions about physical reality, while Walker's proof is more effective on a conceptual level since it leads to the equation of parabolic motion"*.

We note that for the student mathematics is mentioned only by referring to the parabola, not for example by referring to Euclid's theory of proportions. Another mathematics student: *"Galileo in fact chooses a sensory-visual approach, while in Walker we proceed in an abstract way and the demonstrative structure is more explicit"*.

And even a physics student uses similar terms: *"Galileo's proof is characterized by being more "sensitive" than Walker's one: it is as if Galileo's proof had been built step by step by defining the tools and assumptions that uses. Galileo's demonstration is comparable to a recipe where the basic ingredients are introduced and described and then how to work with them is explained, giving precise instructions that lead to the final result"*.

Several students misinterpreted a paradox presented during the lesson on argumentation and proof, perhaps not sufficiently discussed by the teacher, who spoke of intuitive proofs and rigorous proofs. Several students expressed like there were only two types of proof: those intuitive but not rigorous and those rigorous but too abstract, those so simple that they seem superfluous or those too difficult that you don't understand.

For example a mathematics student wrote: *"In the didactic field there are two fundamental types of use of thought: one more intuitive, immediate, self-evident and clear, one trusts the validity spontaneously without rigorous affirmation of concatenations, the other more meditated, analytical, which often requires to think about it. If the result is intuitive, the demonstration has no convincing power, in the second case the passages are seen only as something formal, they create difficulties."*

We observe that not all the students do seem aware that mathematics is a part of physics, thinking that there are mathematical proofs and physical proofs. It is also interesting to note the prejudice that algebra is more mathematical than geometry, which appears closer to empirical reality.

Lack of an introduction to the concept of proof Before answering the questionnaire we have already discussed, 42 students on the course responded to an anonymous interview on wooclap.

To the question: before this course, have you ever thought deeply about the concept of proof? Only 1 student chose as an answer: *"yes, explicitly and in depth"*. 14 chose as an answer: *"yes, explicitly but not in depth"*, 16 chose: *"a little, but not in a systematic way"* a little, but never in a systematic way "and 11 chose: *" no, not explicitly"*. Which means in this case most of the students, 64 percent, even if they follow a physics-mathematical course of study, have never thought about the concept of proof.

This may explain why, as emerged from the question *"which of the two demonstrations of parbolic motion did you find more convincing?"*, nearly 70 percent of students said they found Walker's one more convincing. We might think this is due to Walker's proof being taken from a textbook and more aligned with contemporary teaching. But from the answers given by the students in their paper, we have seen a difficulty emerge in recognizing Galileo's proof as a legitimate mathematical proof.

Conclusions

My interest in the interdisciplinarity between mathematics and physics stems from the simple observation that these are subjects often taught by the same teacher in high school; however in university courses the correlation between these two disciplines from a didactic point of view is not stressed. The growing interest the field of interdisciplinarity between mathematics and phyisics is underlined by various factors, among which we mention the birth of STEM education and the reform of the Italian state exam which concludes the path of the scientific high school, where the classic written test in mathematics was from 2019 converted into a math and physics test.

The choice to start the thesis with a chapter of historical introduction was spontaneous but not accidental. The construction of the IDENTITIES didactic module also reflects this choice. In my opinion, the lack of historicization creates an unnatural separation in the didactic sphere. In a context where everything is "history": history of literature, history of art, history of philosophy, history tout-court, scientific subjects, except for a few brief hints that are considered unimportant, are often presented either or at least perceived as revealed truths, abstract and disconnected from the rest of knowledge.
At the suggestion of Professor Coen from Bologna, I was studying Chemla's text *The history of mathematical proof in ancient traditions* (2012), I was lucky enough to find in the mathematics library of Padua the small text by Swetz and Kao, *Was Pythagoras Chinese?* (1977). I found it particularly interesting to read there a proof of the Pythagoras' theorem, much more intuitive and no less consistent than the one presented in Euclid's elements some two hundred years later. Having the opportunity to reflect on historical information is something that transforms the perception of a discipilina, makes it something human and alive.

The historical discourse integrates well with the teaching theory papers that we have analyzed. The more general objective is to make the object of study something that can be questioned and that can give different answers depending on the questions that are asked. Treating mathematics, which is part of physics, as a language through which reasoning can be built, is useful for a deeper learning of physics; it is also something that restores dignity to mathematics, commonly perceived as "that thing you count with".

The specific idea for the thesis was proposed to me by Professor Levrini, who allowed me to follow the IDENTITIES module live at the University of Bologna,

before I moved to Trento.
It was interesting for me to understand how a research study in the didactic field is carried out, and how many nuances should be considered. Analyzing the students' responses to the module, although the majority were positive, and some of these also quite precise and punctual, we saw that several students had difficulties, more than one might expect in degree courses in mathematics and physics. it is legitimate to ask whether there are structural problems behind these shortcomings. The fact that 64 percent of students had never systematically addressed the concept of demonstration during their course of study suggests that there is a lack in this sense, especially in the curricula that train new teachers.

Chapter 5

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